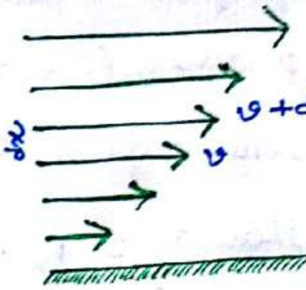
VISCOSITY



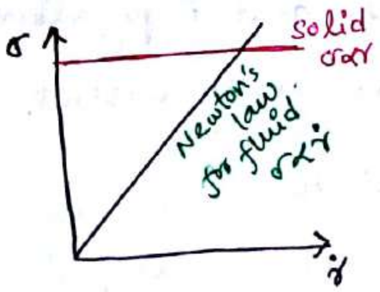
The central property that distinguishes a fluid from a solid is that, a fluid cannot sustain a "shear stress" for any period of time, meaning if a shear is applied to a fluid, it will move under the shear. Thicker liquids like paint, honey, cornstarch solution move less easily than fluids like water, glycerol or air. The measure of the ease with which a fluid yields is its viscosity.

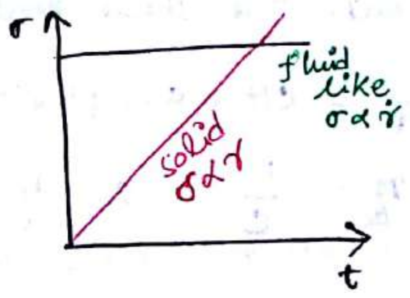
When a liquid is at rest, we do not observe any rigidity or shape elasticity in it but when the liquid is in orderly motion (not turbulent, but streamline), there comes into play a tangential stress between any two layers of the fluid, that are moving relative to each other. Difference in velocities between these two layers gives rise to internal friction, as a result of which the faster layer tends to accelerate the slower one & vice versa.

Newton found that for a fluid moving in parallel layers, the shearing stress at any point where the velocity gradient $\frac{\partial v_x}{\partial y}$ perpendicular to the direction of motion, the frictional force F_x is proportional to area of fluid layer & $\frac{\partial v_x}{\partial y}$.

$$\therefore \; F_x \propto A \qquad \therefore \; F_x = \eta A \frac{\partial v_x}{\partial y}$$

$$\propto \frac{\partial v_x}{\partial y} \qquad \text{or} \quad \frac{F_x}{A} = \eta \frac{\partial v_x}{\partial y} \qquad \text{or} \quad \underbrace{\sigma}_{\text{stress}} = \underbrace{\eta}_{\text{coefficient of viscosity}} \underbrace{\dot{\gamma}}_{\text{strain rate}}$$





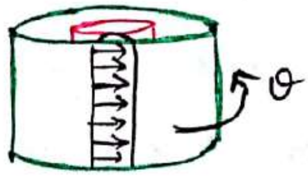
This is Newton's law of viscosity and fluid exhibiting such phenomena is called Newtonian fluid.

Compare with Hooke's law of elastic solid where stress $\propto$ strain ($\sigma = Y\gamma$, Y = Young's modulus) while for fluid stress $\propto$ strainrate ($\sigma = \eta\dot{\gamma}$, η = Coefficient of viscosity). If $A = 1$, $\frac{\partial v_y}{\partial x} = 1$ then $\eta = F_x$. So, coefficient of viscosity of a fluid is the tangential stress per unit velocity gradient. C.G.S. unit of η is "Poise". 1 Poise η means that a force of 1 dyne is required to maintain a velocity difference of 1 cm/sec between two layers of 1 square cm area which are 1 cm apart from each other.

Dimension of η: $\eta = \dfrac{F/A}{\frac{\partial v_y}{\partial x}}$, $\therefore [\eta] = \dfrac{[MLT^{-2}]/[L]^2}{[LT^{-1}]/[L]}$

$$= [ML^{-1}T^{-1}]$$

Fugitive elasticity:



other than sliding planes geometry, Couette flow can be generated by sandwiching a liquid between two concentric cylinders with inner (outer) one stationary. From consideration of shape elasticity, tangential stress $\sigma = n\theta$ where n = modulus of rigidity and θ is the angle of shear.

In the limit $\lim_{\theta \to 0}$, $\sigma = n \tan\theta = n\frac{dy}{dx}$

from Newton's law of viscosity, $\sigma = \eta\frac{dv}{dx} = \eta\frac{d}{dx}\left(\frac{dy}{dt}\right)$

$$= \eta\frac{d}{dt}\left(\frac{dy}{dx}\right) = \eta\frac{d\theta}{dt}.$$

$\frac{d\theta}{dt}$ represents the rate at which the shear brakes and is proportional to the angle of shear. $\therefore \frac{d\theta}{dt} = c\theta$, c = proportionality constant.

$\therefore \sigma = \eta c\theta = n\theta$ or $\eta = \frac{n}{c}$ $\therefore \eta \propto n$

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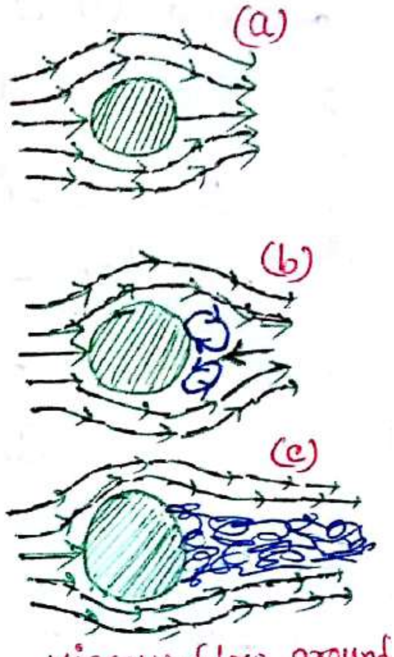
$\frac{1}{c}$ is the relaxation time of the medium fluid that measures the time taken by the shear to disappear, if force is taken off.

So the appearance of viscous force during streamline motion of a liquid is due to the existence of intermittent shear elasticity (fugitive elasticity). Maxwell stated that viscosity in a fluid is due to the existence of fugitive elasticity in it.

Streamline & Turbulent motion - Critical velocity & Reynold's number :



Viscous flow around a ~~cird~~ circular cylinder

If fluid flow is such that magnitude & direction of velocity at any point is always same then its called a steady streamline flow. As in panel (a), no two streamlines can cross each other & the tangent to a line at any point gives the flow direction. In turbulent flow as panel (b) & (c) velocity magnitude & direction changes in irregular manner in terms of eddies, vortices, zigzag motion.

A difference of pressure is maintained between ends for flow of liquid through a tube (Poiseuille's flow). The layer of liquid in contact with the wall of the tube is at rest ("no-slip" condition where both normal & tangential component of velocity is zero). Velocity of the layer increases towards the axis of the tube. The streamline flow is maintained when the velocity is below a certain limit known as "Critical velocity."

When critical velocity is reached, indifferent parts of the liquid no layer travel in a straight line along the tube & when velocity is further increased, streamline motion is completely lost (turbulent).

Reynolds using dimensional analysis showed that in Poiseuille flow, critical velocity v is related to the fluid density ρ, radius of tube r & coefficient of viscosity η.

Suppose, $v = Re\,\eta^x \rho^y r^z$ where Re = Reynold's number whose value is 1000 for narrow tubes. In general, for a liquid of high viscosity even for high velocity streamline motion is observed while for high density & wide bore of tube makes the motion turbulent.

Substituting dimensions, we have

$$[LT^{-1}] = [ML^{-1}T^{-1}]^x [ML^{-3}]^y [L]^z$$

Re is dimensionless number

$$= [M^{x+y}][L^{-x-3y+z}][T^{-x}]$$

Equating the powers of $[M]$, $[L]$ and $[T]$, we have

$$\left.\begin{aligned} x + y &= 0 \\ -x - 3y + z &= 1 \\ -x &= -1 \end{aligned}\right\} \quad x = 1,\ y = -1,\ z = -1$$

$$\therefore\ v = Re\,\eta\rho^{-1}r^{-1}$$

or $\boxed{Re = \dfrac{\rho v r}{\eta}}$

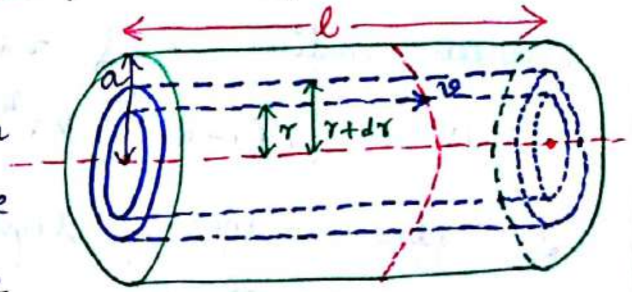
While discussing on motion of dry water, we will see that "vorticity" $\vec{\Omega} = \vec{\nabla} \times \vec{v}$ follows a simple kinetic equation

$$\frac{\partial \vec{\Omega}}{\partial t} + \vec{\nabla} \times (\vec{\Omega} \times \vec{v}) = \frac{1}{Re} \nabla^2 \vec{\Omega}.$$

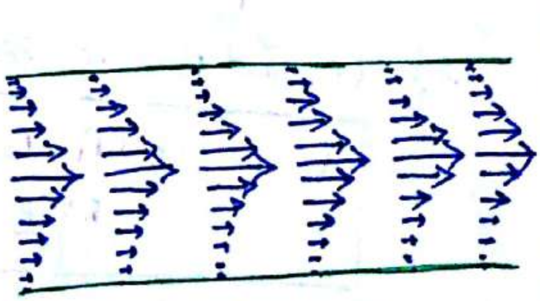
This means that if we solve the flow problem for v_1 for a certain cylinder with radius r_1 & then ask about flow for a different radius r_2 for a different fluid with velocity v_2, the Reynold's number will be same means flows will appear same.

$\therefore Re = \frac{\rho_1 v_1 r_1}{\eta_1} = \frac{\rho_2 v_2 r_2}{\eta_2}$. So we can determine the flow of air past an airplane wing without building an airplane to try but instead make a model with velocity to yield same Re. We can only apply so provided we are dealing with "incompressible" liquid & not with "compressible" gas. Otherwise speed of sound in terms of "Mach number" has to be taken into account. $Ma = \frac{\text{speed of sound in fluid}}{\text{speed of sound in air}}$. So for velocities near the speed of sound or above, the flows are the same in two situations if both "Ma" & "Re" are same for both situations.

Poiseuille's equation for flow of liquid through a horizontal narrow tube:



Consider a horizontal streamline motion of a liquid through a narrow tube. The lines are parallel to the axis of tube (no radial flow). Pressure varies along the length of tube & due to no-slip, velocity of the liquid gradually decreases radially from the axis towards the wall of the tube.



When steady state flow is attained, let v be the velocity at a distance r from the axis of tube & velocity gradient is $\frac{dv}{dr}$. So the tangential stress is $\eta \frac{dv}{dr}$. This force acts over the unit area of surface of cylinder at r in a direction opposite to the pressure gradient. So the total resisting force over the surface of the liquid cylinder is $2\pi r l \, \eta \frac{dv}{dr}$.

If P is the pressure difference between the ends of the tube, then the active force is $P\pi r^2$ (As $P = \frac{F}{A} = \frac{F}{\pi r^2}$). This force tends to accelerate the liquid in cylinder and therefore in steady state, this accelerating force is balanced by the viscous retarding force,

$$P\pi r^2 = -\eta \frac{dv}{dr} 2\pi r l \quad \left(\because \frac{dv}{dr} < 0 \text{ as } v \text{ decrease with increasing } r \right)$$

$$\text{or} \quad r\,dr = -\frac{2\eta l}{P} dv$$

Integrating with the Boundary condition (B.C.) $v = 0$ at $r = a$,

$$\int_r^a r\,dr = -\frac{2\eta l}{P} \int_v^0 dv \Rightarrow \boxed{v = \frac{P}{4\eta l}(a^2 - r^2)}$$

This is an equation for parabola. Now if dV is the volume of liquid that flows through the cylindrical shell per unit time between radius r & $r+dr$ then

$$dV = \left[\pi(r+dr)^2 - \pi r^2\right] v = 2\pi r\,dr\,v = \frac{\pi P}{2\eta l}(a^2 - r^2)\,r\,dr$$

$\therefore$ Total volume of liquid passing through the tube per unit time is

$$V = \frac{\pi P}{2\eta l}\int_0^a (a^2 - r^2)\,r\,dr = \frac{\pi P}{2\eta l}\left(a^2\int_0^a r\,dr - \int_0^a r^3\,dr\right)$$

$$= \frac{\pi P}{2\eta l}\left(\frac{a^4}{2} - \frac{a^4}{4}\right) = \frac{\pi P a^4}{8\eta l}$$

If P_1 and P_2 are the pressure at two ends of the tube then

$$\boxed{V = \frac{\pi(P_1 - P_2)a^4}{8\eta l}}$$

This is known as Poiseuille's equation

<u>Correction to Poiseuille's formula</u>:

Poiseuille's equation $V = \frac{\pi P a^4}{8\eta l}$ is approximately true because two important factors are not taken into account. (a) pressure difference P is utilized partly in communicating kinetic energy to the liquid (b) acceleration of the liquid along the axis of the tube is neglected.

A finite acceleration at inlet of the tube becomes zero only after traversing a finite distance, so 1.64a is added to the length l. To find the kinetic energy correction, let's consider that p is the effective pressure difference that overcomes viscosity.

The workdone against viscous force per unit time is pV & Kinetic energy per unit time is $= \int_0^a \frac{1}{2}\underbrace{\rho\, 2\pi r dr\, v}_{mass}.\, v^2$

$$= \int_0^a \rho\pi v^3 r dr = \int_0^a \rho\pi\left(\frac{p}{4\eta l}\right)^3 (a^2-r^2)^3 r dr = \rho\pi\left(\frac{p}{4\eta l}\right)^3 \int_0^a (a^2-r^2)^3 r dr$$

$$= \rho\pi\left(\frac{p}{4\eta l}\right)^3 \frac{a^8}{8}.$$ So total loss of energy $= pV + \rho\pi\left(\frac{p}{4\eta l}\right)^3 \frac{a^8}{8}$

and this is equal to PV

$$\therefore\quad pV + \rho\pi\left(\frac{p}{4\eta l}\right)^3 \frac{a^8}{8} = PV$$

$$\text{or}\quad pV + \frac{\rho}{\pi^2 a^4}\left(\frac{\pi p a^4}{8\eta l}\right)^3 = PV$$

$$\text{or}\quad pV + \frac{\rho V^3}{\pi^2 a^4} = PV \qquad \left(\because V = \frac{\pi p a^4}{8\eta l}\right)$$

$$\text{or}\quad p = P - \frac{\rho V^2}{\pi^2 a^4}.$$

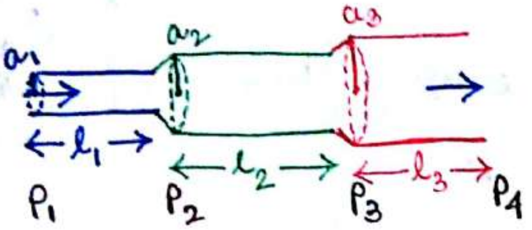
Experimentally it was found by Hagenbach, Couette & Wilberforce that correct form is $p = P - \frac{k\rho V^2}{\pi^2 a^4}$ with $k \approx 1$, differing for different scenario.

Corrected Poiseuille's formula is

$$\boxed{V = \frac{\pi a^4}{8\eta(l + 1.64a)}\left(P - \frac{k\rho V^2}{\pi^2 a^4}\right)}$$

Flow of liquid through capillaries in series & parallel

Poiseuille's formula $V = \frac{P}{8\eta l/\pi a^4} = \frac{P}{R}$ can be compared with Ohm's law for flow of electric current through a resistance $i = \frac{E}{R}$. So the rate of flow of the liquid V corresponds to current i, pressure difference P to the potential difference E and $\frac{8\eta l}{\pi a^4}$ to the resistance R.



Consider a series connection of three capillaries with radius a_1, a_2, a_3 & length l_1, l_2, l_3. Let P_1 & P_4 are the pressure at extreme ends & P_2, P_3 are pressure at the junctions. As there is no accumulation of the liquid at the junction, so V must be equal through all capillaries, just like current is same through any resistance connected in series.

$$\text{So,} \quad V = \frac{\pi(P_1 - P_2)a_1^4}{8\eta l_1} = \frac{\pi(P_2 - P_3)a_2^4}{8\eta l_2} = \frac{\pi(P_3 - P_4)a_3^4}{8\eta l_3}$$

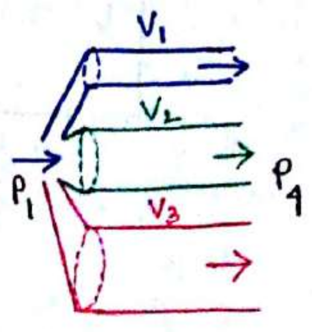
$$\therefore \left.\begin{aligned} P_1 - P_2 &= \frac{8\eta l_1 V}{\pi a_1^4} \\ P_2 - P_3 &= \frac{8\eta l_2 V}{\pi a_2^4} \\ P_3 - P_4 &= \frac{8\eta l_3 V}{\pi a_3^4} \end{aligned}\right\}$$

$$\text{Summing,} \quad P_1 - P_4 = \frac{8\eta V}{\pi}\left[\frac{l_1}{a_1^4} + \frac{l_2}{a_2^4} + \frac{l_3}{a_3^4}\right]$$

$$\text{or} \quad V = \frac{\pi P}{8\eta}\left[\frac{l_1}{a_1^4} + \frac{l_2}{a_2^4} + \frac{l_3}{a_3^4}\right]^{-1} \quad \text{—— (1)}$$

where $P = P_1 - P_4$ is pressure difference across composite slab.

If we maintain $P = P_1 - P_4$ across ends of three capillaries connected in parallel, then volume of liquid flowing per unit time through them is



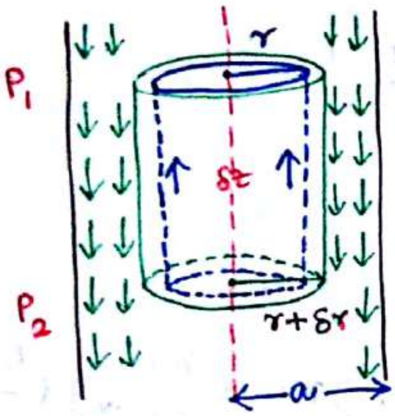
$$V = V_1 + V_2 + V_3 = \frac{\pi P a_1^4}{8\eta l_1} + \frac{\pi P a_2^4}{8\eta l_2} + \frac{\pi P a_3^4}{8\eta l_3}$$

$$V = \frac{\pi P}{8\eta}\left(\frac{a_1^4}{l_1} + \frac{a_2^4}{l_2} + \frac{a_3^4}{l_3}\right) = P\left(\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}\right) \quad (2)$$

Comparing (1) and (2) we recover the effective resistance for an equivalent series connected viscous flow as $R = R_1 + R_2 + R_3$ while for parallel connected, $\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}$ where

$$R_1 = \frac{8\eta l_1}{\pi a_1^4}, \quad R_2 = \frac{8\eta l_2}{\pi a_2^4}, \quad R_3 = \frac{8\eta l_3}{\pi a_3^4}.$$

Downward flow of a liquid through a vertical narrow tube:



Consider a vertical narrow tube through which a liquid flows steadily. Consider a cylindrical shell with radius r & $r+\delta r$ and length δz. So the viscous force on the inner shell wall in vertical upward direction is $2\pi r\,\delta z\,\eta \frac{dv}{dr}$.

Viscous force on the outer wall in vertical downward direction is $2\pi r\,\delta z\,\eta\frac{dv}{dr} + \frac{d}{dr}\left(2\pi r\,\delta z\,\eta \frac{dv}{dr}\right)\delta r$

Force due to pressure p on the upper annular flat surface of the element of the liquid cylinder in the vertical downward direction is $2\pi r\,\delta r\, p$ & similarly force due to pressure on the lower annular flat surface of the liquid cylinder in the upward direction is $2\pi r\,\delta r\left(p + \frac{dp}{dz}\delta z\right)$

At steady state, liquid acceleration $= 0$, resultant downward force is zero.

$$\therefore -\cancel{2\pi r\,\delta z\,\eta\frac{dv}{dr}} + \cancel{2\pi r\,\delta z\,\eta\frac{dv}{dr}} + \frac{d}{dr}\left(2\pi r\,\delta z\,\eta\frac{dv}{dr}\right)\delta r + \underbrace{2\pi r\,\delta r\,\delta z\,\rho g}_{\text{[vertical downward force due to weight of cylindrical shell]}}$$

$$+ \cancel{2\pi r\,\delta r\, p} - 2\pi r\,\delta r\left(\cancel{p} + \frac{dp}{dz}\delta z\right) = 0$$

[vertical downward force due to weight of cylindrical shell]

$$\text{or}\quad 2\pi\,\delta z\,\eta\frac{d}{dr}\left(r\frac{dv}{dr}\right) + 2\pi r\,\delta r\,\delta z\,\rho g - 2\pi r\,\delta r\frac{dp}{dz}\delta z = 0$$

or, $\eta \frac{d}{dr}\left(r\frac{dv}{dr}\right) + r\rho g = r\frac{dp}{dz}$ $\quad [\because \eta \neq \eta(r)]$

or $\frac{dp}{dz} = \frac{\eta}{r}\frac{d}{dr}\left(r\frac{dv}{dr}\right) + \rho g.$ or $-\frac{P_1-P_2}{l} = \frac{\eta}{r}\frac{d}{dr}\left(r\frac{dv}{dr}\right)+\rho g$

or $\left(\rho g + \frac{P_1-P_2}{l}\right) r = -\eta \frac{d}{dr}\left(r\frac{dv}{dr}\right)$

Integrating with B.C. $r=a,\ v=0$

$$\left(\rho g + \frac{P_1-P_2}{l}\right)\frac{r^2}{2} + A = -\eta r \frac{dv}{dr}$$

Integrating once again, $\left(\rho g + \frac{P_1-P_2}{l}\right)\frac{r^2}{4} + A\ln r + B = -\eta v$

at $r=0$, $v \neq \infty$ (remember $\ln 0 = -\infty$), A must be zero. &

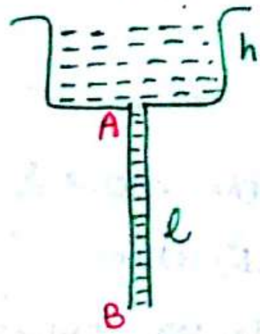
substituting $r=a,\ v=0$. $\quad B = -\left(\rho g + \frac{P_1-P_2}{l}\right)\frac{a^2}{4}$

$$\therefore\ \eta v = \left(\rho g + \frac{P_1-P_2}{l}\right)\left(\frac{a^2-r^2}{4}\right)$$

So the volume of liquid flowing per unit time is

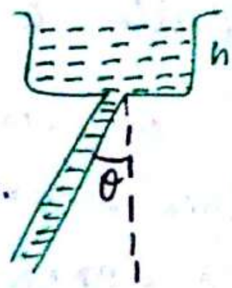
$$V = \int_0^a 2\pi r\,dr\,v = \frac{\pi}{2\eta}\left(\rho g + \frac{P_1-P_2}{l}\right)\int_0^a (a^2-r^2)\,r\,dr = \frac{\pi a^4}{8\eta}\left(\rho g + \frac{P_1-P_2}{l}\right)$$

In liquid viscometer, a narrow tube is connected to a liquid container. $P_1 = \pi + \rho g h$ is the pressure at inlet A and $P_2 = \pi$ is the pressure at outlet B where π is the barometric pressure.



$$\therefore\ V = \frac{\pi a^4}{8\eta}\left(\rho g + \frac{\rho g h}{l}\right) = \boxed{\frac{\pi a^4}{8\eta l}\rho g(l+h)}$$

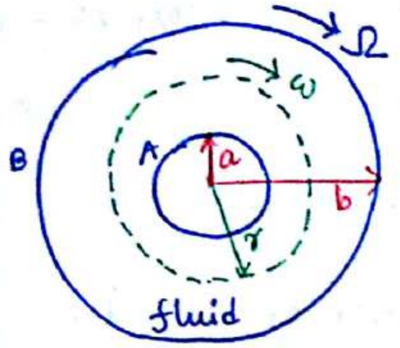
For inclined liquid viscometer, acceleration due to gravity g has $g\cos\theta$ component along vertical direction.



$$\text{So, } V = \frac{\pi a^4}{8\eta}\left(\rho g\cos\theta + \frac{P_1-P_2}{l}\right) = \frac{\pi a^4}{8\eta}\left(\rho g\cos\theta + \frac{\rho g h}{l}\right)$$

$$= \boxed{\frac{\pi a^4}{8\eta l}\rho g(l\cos\theta + h)}$$

Torque on a cylinder immersed in a rotating fluid



A viscous liquid is filled within two coaxial cylinders A & B with cylinder B rotating about common axis with constant angular velocity Ω. The torque on A due to rotation B is required. The innermost layer of fluid at A is zero velocity (no-slip), a, b, l are the radii of A & B cylinder and length. Suppose the fluid at a distance r rotates with angular velocity ω. Its linear velocity is $r\omega$ and velocity gradient is $\frac{d}{dr}(r\omega) = r\frac{d\omega}{dr} + \cancel{\omega}^{\to 0}$ ↳ no viscosity effect.

∴ Viscous force on the side of fluid cylinder is

$F = \eta\, 2\pi r l\, r\frac{d\omega}{dr}$ & then the viscous torque $\Gamma = F \cdot r$
$= 2\pi\eta l r^3 \frac{d\omega}{dr}$

As the fluid rotates in steady state, torque on inner cylinder is in clockwise direction

$\Gamma.$ $\quad$ or $\Gamma\frac{dr}{r^3} = 2\pi\eta l\, d\omega$

Integrating, $\Gamma\int_a^b \frac{dr}{r^3} = 2\pi\eta l\int_0^\Omega d\omega$

or $\frac{\Gamma}{2}\left(\frac{1}{a^2} - \frac{1}{b^2}\right) = 2\pi\eta l\Omega$ $\quad$ or $\boxed{\Gamma = \frac{4\pi\eta l\Omega a^2 b^2}{b^2 - a^2}}$

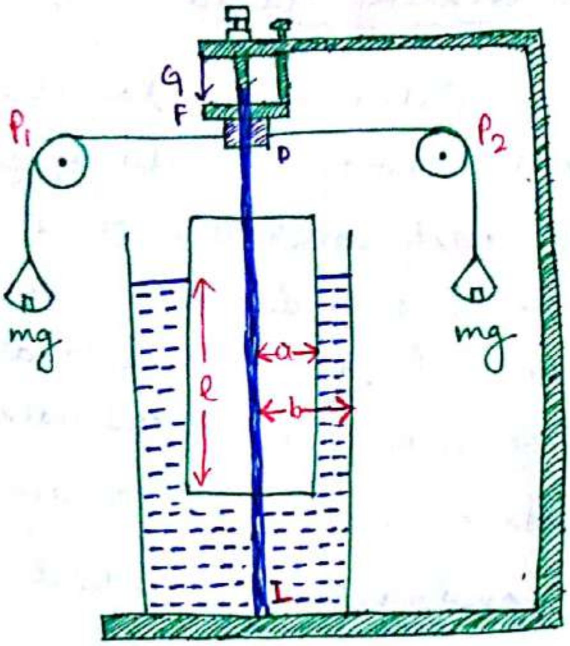
\# Suppose now A rotates with Ω & Γ is the clockwise torque

$\Gamma = -2\pi\eta l r^3\frac{d\omega}{dr}$ $\quad$ ($\Gamma < 0$ as $\frac{d\omega}{dr} < 0$ as ω decreases with increasing r)

∴ $-\Gamma\int_a^b\frac{dr}{r^3} = 2\pi\eta l\int_\Omega^0 d\omega$ $\quad$ or $\frac{\Gamma}{2}\left(\frac{1}{b^2} - \frac{1}{a^2}\right) = -2\pi\eta l\Omega$

or $\boxed{\Gamma = \frac{4\pi\eta l\Omega a^2 b^2}{b^2 - a^2}}$

Searle's viscometer uses this technique for measurement of η of highly viscous liquids. Two weights rotate the inner



Searle's viscometer

cylinder by ball-bearing pulleys while the outer cylinder is fixed with the fluid in between. We know

$$\Gamma = \frac{4\pi\eta l \Omega a^2 b^2}{b^2 - a^2} = mgd$$

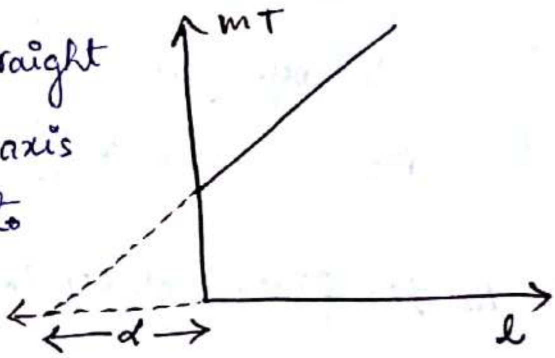
where d = diameter of drum D.

$$\text{so } \eta = \frac{gd\,(b^2-a^2)}{4\pi\Omega a^2 b^2}\,\frac{m}{l} \quad \text{and as}$$

$T = \dfrac{2\pi}{\Omega}$ is the time period of rotation

$$\eta = \frac{gd\,(b^2-a^2)}{8\pi^2 a^2 b^2}\,\frac{mT}{l}.$$

For a given liquid, $\dfrac{mT}{l}$ = constant



Though it should be a origin-passing straight line, but experimentally found to cut Y axis so that the above expression is modified to accomodate

$$\eta = \frac{gd\,(b^2-a^2)}{8\pi^2 a^2 b^2}\,\frac{mT}{l+\alpha}$$

Viscosity of high viscous liquids

Stoke's law : Viscous resisting force on a small sphere falling through a liquid of infinite extent is $F = 6\pi\eta a v$ where v = terminal velocity of the sphere, a = radius.

viscous retarding force = effective gravitational force.

$$\frac{4}{3}\pi a^3(\rho-\sigma)g = 6\pi\eta a v \,, \quad \begin{cases} \rho = \text{density of the sphere} \\ \sigma = \text{density of the liquid.} \end{cases}$$

$$\text{or } \eta = \frac{2}{9}\,\frac{a^2(\rho-\sigma)g}{v}$$

In practice in cylindrical vessel due to confinement, boundary effect due to wall & bottom of cylinder is corrected to yield

$$\eta = \frac{2}{9}\frac{a^2(\rho-\sigma)g}{v\left(1+2.4\frac{a}{R}\right)\left(1+3.3\frac{a}{h}\right)}$$; R = radius of cylinder, h = height of liquid.

Using dimensional analysis $F = 6\pi\eta a v$ can be deduced as

$F = K a^x \eta^y v^z$, K = dimensionless number.

$\therefore [MLT^{-2}] = [L]^x [ML^{-1}T^{-1}]^y [LT^{-1}]^z = M^y L^{x-y+z} T^{-y-z}$

Equating the powers of M, L, T, $y=1$, $x-y+z=1$, $-y-z=-2$

$\therefore x=y=z=1 \quad \therefore F = K\eta a v$

By solving $\eta\nabla^2\vec{v} - \vec{\nabla}p = 0$ for a sphere in a liquid with incompressibility constaint $\vec{\nabla}\cdot\vec{v}=0$, Stokes calculated $K = 6\pi$.

$\therefore F = 6\pi\eta a v$.

Observe that from $v = \frac{2}{9}\frac{ga^2(\rho-\sigma)}{\eta}$, $v \propto a^2$. For raindrop of radius $a = 10^{-3}$ cm falling through air with $\eta = 1.8\times10^{-4}$ prise

terminal velocity $v = \dfrac{2\times 981\times 10^{-6}}{9\times 1.8\times 10^{-4}} = 1.2$ cm/sec.

$\rho = 1$, $\sigma\to 0$ for air.

That is why raindrops fall with slowrate. $v \neq f(p)$ as $\eta \neq \eta(p)$. Notice also that its independent of mass.

Bigger raindrops fall rapidly through air as Stoke's law does not hold if $a > 0.01$ cm because of turbulence, where $F \propto \sqrt{v}$.

If $\sigma > \rho$ then $v < 0$ meaning if density of fluid > density of body, then body moves through the fluid in the upward direction. This is the reason, why air bubbles in water or in any other liquid rise up.

Equation of motion of a body falling through a viscous medium

Viscous force $\propto$ velocity (instantaneous), but now we have downward force mg due to gravity for sedimentation & no thermal (Brownian) force.

$$m\frac{d^2x}{dt^2} = mg - f = mg - \gamma\frac{dx}{dt}$$

or $\frac{d}{dt}\left(\frac{dx}{dt}\right) = g - \frac{\gamma}{m}\frac{dx}{dt}$ or $\frac{dv}{dt} = g - \frac{\gamma}{m}v$

or $\frac{dv}{g - \gamma/m\,v} = dt$ or $-\frac{m}{\gamma}\frac{d(g - \gamma/m\,v)}{g - \gamma/m\,v} = dt$

or $\int \frac{d(g - \gamma/m\,v)}{g - \gamma/m\,v} = -\frac{\gamma}{m}\int dt + c$

or $\ln(g - \frac{\gamma}{m}v) = -\frac{\gamma}{m}t + c$. Now substitute the boundary condition, $t = 0$, $v = 0$ $\therefore c = \ln g$.

or $\ln\left(1 - \frac{\gamma v}{mg}\right) = -\gamma t/m$ or $v = \frac{mg}{\gamma}\left(1 - e^{-\frac{\gamma}{m}t}\right)$.

$\simeq \frac{mg}{\gamma}$ (if $\frac{\gamma t}{m} \gg$)

So the maximum (terminal) velocity is $v = mg/\gamma$

Again, $\frac{dx}{dt} = \frac{mg}{\gamma}\left(1 - e^{-\gamma/m\,t}\right)$

Integrating, $x = \frac{mg}{\gamma}\left(t + \frac{m}{\gamma}e^{-\gamma t/m}\right) + c'$. Again substitute the B.C. at $t = 0$, $x = 0$.

$\therefore c' = -\frac{m^2 g}{\gamma^2}$

$\therefore x = \frac{mg}{\gamma}\left(t + \frac{m}{\gamma}e^{-\gamma t/m}\right) - \frac{m^2 g}{\gamma^2}$

$= \frac{mgt}{\gamma} + ge^{-\gamma t/m} - \frac{m^2 g}{\gamma^2}$

Flow of Gas through a narrow tube

Unlike incompressible liquids (density is independent of pressure), gas is compressible (density $\propto$ pressure). So for a liquid, volume flowing through any cross-section in a given time is constant while for a gas, mass flowing through a crosssection in a given time is constant.

$\therefore \rho \propto P$ & $\rho V =$ constant or $PV =$ constant (Boyle's law)

Let us consider an elemental length dx with pressure difference dP within the tube, & is small compared to the tube so that density variation within dx <<<< so that we can still write Poiseuille's equation for flowing liquid

$$V = -\frac{\pi a^4}{8\eta}\frac{dP}{dx}$$. It's –ive because $\frac{dP}{dx} < 0$ so that $V > 0$

If P_1 & P_2 are pressure of gas at inlet & outlet end with V_1 volume entering per unit time, then

$$P_1 V_1 = PV = -P\frac{\pi a^4}{8\eta}\frac{dP}{dx}$$

or $$\int_0^l P_1 V_1\, dx = -\frac{\pi a^4}{8\eta}\int_{P_1}^{P_2} P\,dP = -\frac{\pi a^4}{8\eta}\frac{P_2^2 - P_1^2}{2}$$

or $$P_1 V_1 l = \frac{\pi a^4 (P_1^2 - P_2^2)}{16\eta}$$ or $$P_1 V_1 = \frac{\pi a^4 (P_1^2 - P_2^2)}{16\eta l}$$

and $$\eta = \frac{\pi a^4 (P_1^2 - P_2^2)}{16 P_1 V_1 l}.$$ Here we have assumed "no-slip" or no relative motion between tube wall & adjacent gas layers, which breaks down at low pressure. The corrected form is

$$\eta = \frac{\pi a^4 (P_1^2 - P_2^2)}{16 P_1 V_1 l}\left(1 + \frac{4\lambda}{a}\right) - \frac{\rho V_1}{8\pi l}\left(K + \ln P_1/P_2\right)$$

$\lambda =$ slipping coefficient (constant for gas) & K depends on apparatus.

Dependence of Viscosity on pressure & temperature

Liquids : η of liquids increases rapidly with pressure. However for water, glycerol, η decreases with pressure. η decreases rapidly with increase in temperature. For pure liquids $\eta = \frac{A}{(1+BT)^n}$ with A, B, n depending on nature of liquid. For liquid mixture there is no one recipe.

Gases : Using kinetic theory of gases & experiment, established is

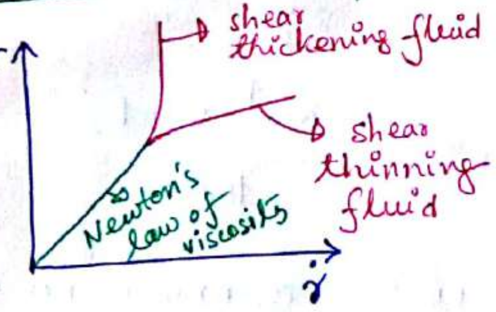
(i) At high pressures, η increases with pressure increment.
(ii) At moderate pressures, η is independent of pressure.
(iii) At low pressures, $\eta \propto P$.

(i) At high temperature, η rapidly increases with temperature & for Mercury $\eta = \alpha T^{1.6}$ (Kinetic theory $\eta = KT^{0.5}$).
(ii) At moderate temperature, $\eta = \eta_0 \frac{\alpha T^{0.5}}{1+\frac{S}{T}}$, η_0 is at $0^\circ C$

α, S = constants

(iii) At low temperature, no one formula agrees well.

Non-Newtonian liquids

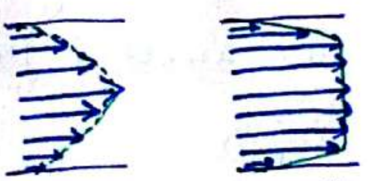


$\sigma \not\propto \dot{\gamma}$ thixotrophic fluid

sol $\leftrightarrow$ gel transition, both property of solid & fluid $\rightarrow$ Glass!!

Examples : shear thinning $\rightarrow$ "colloid"
e.g. paint, milk, blood.

Shear thickening : cornstarch solvent, molten chocolate.



Newtonian non-Newtonian

Displacement of fluid particles are usually non-affine even if Couette flow is established with linear flow profile.

Fluid Motion

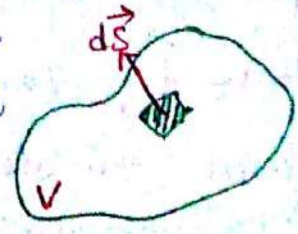
Ideal Fluids – The flow of "dry" Water

Fluid dynamics concerns with the motion of fluids (liquid & gas) in a macroscopic sense to regard as a continuous medium. Infinitely small elements of volume – fluid particle & point in a fluid means very small compared to volume of body but large compared to the molecular distance.

The Equation of Continuity: Mathematically, the state of a moving fluid is given by the fluid velocity distribution over space & time, $\vec{v} = \vec{v}(x, y, z, t)$ and of any two thermodynamic quantities, say pressure $p(x, y, z, t)$ and density $\rho(x, y, z, t)$. So if given 3-components of velocity, pressure & density, state of the fluid is completely determined. Additionally, a conducting fluid will carry an electric current whose density $\vec{j} = \vec{j}(x, y, z, t)$. Similarly temperature or Magnetic field have similar effect.

We neglect first EM field, temperature variation & assume that density is constant or variation in pressure is very small (or the fluid is incompressible). So if the flow velocity is much less than the speed of sound wave in the fluid, density variation can be neglected. $\boxed{\rho = \text{constant}}$

Conservation of Mass: If matter flows away then there must be decrease in the amount of matter left behind. The mass of fluid flowing in unit time through a surface element $d\vec{S}$ bounding the volume is $\rho\vec{v}\cdot d\vec{S}$ & its positive if flowing out (negative otherwise), so that the total mass is $\oint_S \rho\vec{v}\cdot d\vec{S}$



Decrease in fluid mass per unit time is $-\frac{\partial}{\partial t}\int_V \rho\, dV$.

Therefore $\frac{\partial}{\partial t}\int_V \rho\, dV = -\oint_S \rho\vec{v}\cdot d\vec{S} = \int \vec{\nabla}\cdot(\rho\vec{v})\, dV$ using Green's theorem. $\therefore \int_V \left[\frac{\partial \rho}{\partial t} + \vec{\nabla}\cdot(\rho\vec{v})\right] dV = 0$. Since this equation must hold for any volume V, the integrand must vanish. So $\boxed{\frac{\partial \rho}{\partial t} + \vec{\nabla}\cdot(\rho\vec{v}) = 0}$ $\vec{j} = \rho\vec{v}$ = mass flux density.

This is the hydrodynamic equation of continuity leading to conservation of mass. For incompressible fluid ρ = constant & so $\vec{\nabla}\cdot\vec{v} = 0$. Like Magnetic field $\vec{B}$, fluid velocity has zero divergence.

Euler's equation of motion: Change of velocity due to forces, torques, so that Newton's 2^{nd} law become,

Rate of increase of momentum of fluid particle = Sum of forces on fluid particle.

There are 2 types of forces on fluid particles,

- Surface forces $\Rightarrow$ - pressure force, viscous force, gravity force
- Body forces $\Rightarrow$ - centrifugal force, Coriolis force, EM force.

So total force acting on the volume $= -\oint_S P\, d\vec{S} = -\int_V \vec{\nabla}P\, dV$

So fluid surrounding any volume element dV exerts a force $-\vec{\nabla}P\, dV$ or $-\vec{\nabla}P$ per unit volume. There are external forces like electromagnetic, gravity. For conservative force with ϕ = potential per unit mass, $-\rho\vec{\nabla}\phi$ = force density, otherwise for non-conservative force $\vec{f}_{ext}$ has to be taken care. Due to shearing stress in a flowing fluid, there are internal force per unit volume $\vec{f}_{visc}$, so that Newton's law become,

$$\rho \frac{d\vec{v}}{dt} = -\vec{\nabla}p - \rho\vec{\nabla}\phi + \cancel{f_{visc}}$$ dry water/ thin liquid (inviscid flow)

The derivative $\frac{d\vec{v}}{dt}$ denotes not the change of rate of the fluid velocity at a fixed point in space but the rate of change of the velocity of a given fluid particle as it moves about in space. So to express $\frac{d\vec{v}}{dt}$ in terms of quantities referring to fixed in space we see composition from two parts, ① change during dt in the velocity at a fixed point in space, ② velocity difference at same instant at two points $d\vec{r}$ apart.

$$\therefore \rho\frac{d\vec{v}}{dt} = \rho\frac{\partial\vec{v}}{\partial t} + \frac{\rho}{dt}\left(dx\frac{\partial\vec{v}}{\partial x} + dy\frac{\partial\vec{v}}{\partial y} + dz\frac{\partial\vec{v}}{\partial z}\right)$$

$$= \rho\frac{\partial\vec{v}}{\partial t} + \rho\left(\frac{d\vec{r}}{dt}\cdot\vec{\nabla}\right)\vec{v} = \rho\,\partial_t\vec{v} + \rho(\vec{v}\cdot\vec{\nabla})\vec{v}$$

Note that there can be acceleration even though $\frac{\partial\vec{v}}{\partial t} = 0$ so that velocity at a given point is not changing, e.g. water flowing in a circle at constant speed is accelerating due to change in direction of the centripetal acceleration.

$$\therefore \boxed{\frac{\partial\vec{v}}{\partial t} + (\vec{v}\cdot\vec{\nabla})\vec{v} = -\frac{\vec{\nabla}p}{\rho} - \vec{\nabla}\phi}$$ Euler's Equation

If we define $\vec{\Omega} = \vec{\nabla}\times\vec{v}$, then using the vector identity

$$(\vec{v}\cdot\vec{\nabla})\vec{v} = (\vec{\nabla}\times\vec{v})\times\vec{v} + \frac{1}{2}\vec{\nabla}(\vec{v}\cdot\vec{v}) = \vec{\Omega}\times\vec{v} + \frac{1}{2}\vec{\nabla}v^2$$

$$\therefore \frac{\partial\vec{v}}{\partial t} + \vec{\Omega}\times\vec{v} + \frac{1}{2}\vec{\nabla}v^2 = -\frac{\vec{\nabla}p}{\rho} - \vec{\nabla}\phi$$

$\vec{\Omega}$ is called the "**vorticity**" & for an irrotational flow, $|\vec{\Omega}| = 0$

Circulation of a vector field around any arbitrary closed loop in a fluid at a given instant is

Circulation $\Gamma = \oint_c \vec{v}\cdot d\vec{l}$ (line integral)

Circulation Γ $= \int_S \vec{\nabla}\times\vec{v}\cdot d\vec{S}$ (using Stoke's theorem)

$= \int_S \vec{\Omega}\cdot d\vec{S}$. So vorticity $\vec{\Omega}$ is the circulation around a unit area & perpendicular to the direction of $\vec{\Omega}$.

Conservation of Circulation:

Change in circulation around a "fluid contour" moving over space $= \frac{d}{dt}\oint_C \vec{v}\cdot\delta\vec{l} = \oint_C \frac{d\vec{v}}{dt}\cdot\delta\vec{l} + \oint_C \vec{v}\cdot\frac{d\delta\vec{l}}{dt}$

Now $\vec{v}\cdot\frac{d\delta\vec{l}}{dt} = \vec{v}\cdot\delta\frac{d\vec{l}}{dt} = \vec{v}\cdot\delta\vec{v} = \frac{1}{2}\delta(v^2)$ and then

$\oint_C \frac{1}{2}\delta(v^2) = 0$ as total differential along closed contour $= 0$.

$$\therefore \quad \frac{d\Gamma}{dt} = \oint_C \frac{d\vec{v}}{dt}\cdot\delta\vec{l} = \oint_C \left(\frac{\partial\vec{v}}{\partial t} + (\vec{v}\cdot\vec{\nabla})\vec{v}\right)\cdot\delta\vec{l}$$

$$= -\oint_C \vec{\nabla}\left(\frac{P+\phi}{\rho}\right)\cdot\delta\vec{l} \quad \text{(Using Euler's equation)}$$

$$= -\int_S \cancelto{0}{\vec{\nabla}\times\vec{\nabla}}\left(\frac{P+\phi}{\rho}\right)\cdot\delta\vec{S} = 0.$$

$\therefore \oint \vec{v}\cdot d\vec{l} =$ constant (Kelvin's theorem of conservation of circulation)

So for irrotational flow, $\frac{\partial\Omega}{\partial t} = 0$ & so $\vec{\nabla}\cdot\vec{v} = 0$ & $\vec{\nabla}\times\vec{v} = 0$

It is also called "Potential flow". As $\vec{\nabla}\times\vec{v} = 0$ on streamlines, steady flow past any body with a uniform incident flow at infinity must be a potential flow.

Bernoulli's theorem from Euler's equation, taking $\vec{v}\cdot$ operation

$\vec{v}\cdot\vec{\Omega}\times\vec{v} = 0$ & so

$\vec{v}\cdot\vec{\nabla}\left(\frac{P}{\rho} + \phi + \frac{1}{2}v^2\right) = 0$ for steady streamline flow $\frac{\partial v}{\partial t} = 0$.

So for a small displacement in the direction of the fluid velocity

$$\boxed{\frac{p}{\rho} + \phi + \frac{1}{2}v^2 = \text{constant}}$$ for all points along a streamline.

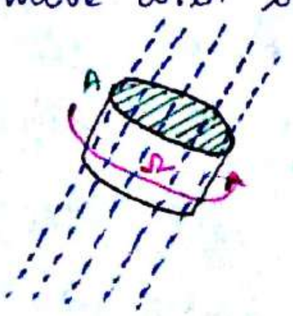
This is called Bernoulli's equation for potential flow. The constant in R.H.S. is constant along any given streamline but is different for different streamlines, while for a potential flow (irrotational), it is constant throughout the fluid.

$$\boxed{\frac{p}{\rho} + \phi + \frac{1}{2}v^2 = \text{constant}}$$ (everywhere).

Vortex lines In terms of vorticity, we have already noted the Euler's equation, $\frac{\partial \vec{v}}{\partial t} + \vec{\Omega} \times \vec{v} + \frac{1}{2}\vec{\nabla} v^2 = -\frac{\vec{\nabla} p}{\rho} - \vec{\nabla}\phi$. By taking a curl, we can permanently eliminate pressure, so that for an incompressible liquid,

$$\boxed{\vec{\nabla}\cdot\vec{v} = 0, \quad \vec{\Omega} = \vec{\nabla}\times\vec{v} \quad \text{and} \quad \frac{\partial \vec{\Omega}}{\partial t} + \vec{\nabla}\times(\vec{\Omega}\times\vec{v}) = 0}$$ describes the velocity field everywhere. Also, if $\vec{\Omega} = 0$ at any time t, $\frac{\partial \vec{\Omega}}{\partial t} = 0$ so at all time $\vec{\Omega} = 0$ or the flow remains permanently irrotational. The equations to be solved are $\vec{\nabla}\cdot\vec{v} = 0$, $\vec{\nabla}\times\vec{v} = 0$

As Helmholtz proposed, imagine that in the fluid we want to draw vortex lines, rather than streamline. Vortex lines are field lines in the direction of $\vec{\Omega}$ and density proportional to the magnitude $|\vec{\Omega}|$. They are similar to magnetic induction $\vec{B}$, without any beginning or end, revolving in closed loops, and move with the fluid. Suppose at time t, a small cylinder of the liquid with axis parallel to vortex lines is transported at $t+dt$ to another cylinder with area A_2.

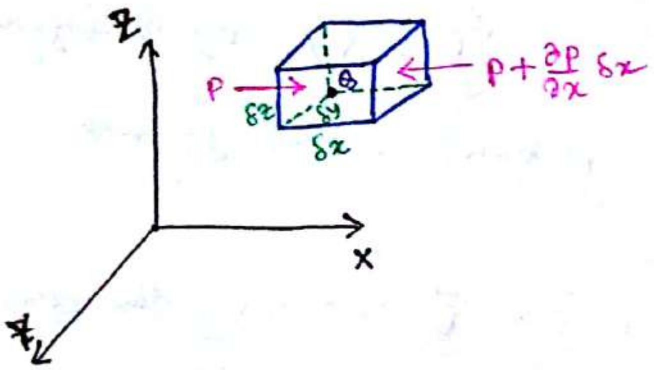


So, $\Omega_1 A_1 = \Omega_2 A_2$ as $\Omega \propto$ density because mass is same in both situation, we see $A_1 = \pi r_1^2$ and $A_2 = \pi r_2^2$ and $M_1 = M_2$ gives

$$\pi M_1 r_1^2 \Omega_1 = \pi M_2 r_2^2 \Omega_2 \quad \text{is} \quad L_1 = L_2$$

or in the absence of viscosity, angular momentum of an element of the fluid is invariant. This is "ideal" "dry" water case as it means if $\vec{\Omega} = 0$ then $\vec{\Omega}$ cannot be created or there will not be any vorticity.

Fluid Statics: Condition of Equilibrium of a fluid



Consider a container of fluid at rest, & within it an infinitesimal rectangular parallelopiped is taken in which at point Q, we calculate the body force. F_x, F_y, F_z are components of the body force $\vec{F}$ at $Q(x, y, z)$. Now, force due to pressure p on the elemental area $\delta y \delta z$ along x-axis is $p\,\delta y \delta z$. & force on opposite face of the parallelopiped is $-(p + \frac{\partial p}{\partial x}\delta x)\delta y \delta z$.

$$\therefore \text{ The resultant force } = p\,\delta y \delta z - (p + \frac{\partial p}{\partial x}\delta x)\delta y \delta z$$
$$= -\frac{\partial p}{\partial x}\delta x \delta y \delta z.$$

So for equilibrium under the action of the body force

$$F_x \rho\, \delta x \delta y \delta z - \frac{\partial p}{\partial x}\delta x \delta y \delta z = 0 \quad \text{or,} \quad F_x = \frac{1}{\rho}\frac{\partial p}{\partial x}$$

Similarly for y and z direction, $F_y = \frac{1}{\rho}\frac{\partial p}{\partial y}$, $F_z = \frac{1}{\rho}\frac{\partial p}{\partial z}$.

$$\therefore \vec{F} = \frac{1}{\rho}\vec{\nabla} p. \quad \text{and for } \rho = \text{constant}, \ \vec{\nabla} \times \vec{F} = 0.$$

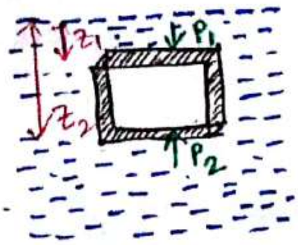
If the force is gravity then $F_z = -g$, $F_x = F_y = 0$.

$$\text{So, } -g - \frac{1}{\rho}\frac{dp}{dz} = 0 \quad \text{is} \quad dp = -g\rho\, dz.$$

$\therefore\; P = -\rho g z + C$ where at $z=0$, $P=P_0$ gives $C=P_0$

$\therefore\; \boxed{P = P_0 - \rho g z}$ Equation of hydrostatics for incompressible liquid

As $P-P_0$ does not depend on P_0 or pressure exerted by external forces on the fluid is transmitted equally in all directions. This is Pascal's principle.



Again, consider a body of cross-sectional area A immersed in a fluid with pressure P_1 & P_2 at upper & lower surface, then

$$P_1 = -\rho g z_1 + C_1,\quad P_2 = -\rho g z_2 + C_2$$

$$\therefore (P_1 - P_2) = \rho g (z_2 - z_1).$$

$$\therefore (P_1 - P_2)A = \text{thrust} = \rho g A(z_2 - z_1) = \text{weight of the fluid dispersed in upward direction.}$$

This is the Archimede's principle.

This can be easily derived from Bernoulli's theorem

$\frac{v^2}{2} + \frac{P}{\rho} + gh = \text{constant}$ by substituting $v=0$, $h=z$.

For compressible gases, Boyle's law give $P \propto \rho$.

$\therefore\; \frac{P}{P_0} = \frac{\rho}{\rho_0}$. So from $dP = -g\rho dz$ we get $dP = -g dz \frac{\rho_0}{P_0} P$

$$\therefore\; \frac{dP}{P} = -\frac{g\rho_0}{P_0} dz \qquad \therefore \int_{P_0}^{P} \frac{dP}{P} = -\frac{\rho_0 g}{P_0}\int_0^z dz$$

$\therefore\; \ln \frac{P}{P_0} = -\frac{\rho_0 g z}{P_0}$ [where P_0 = pressure at surface of Earth or $z=0$.]

$$\therefore\; P = P_0 e^{-\frac{\rho_0 g}{P_0} z}$$

This expression correctly shows exponential fall of pressure with distance but flawed as temperature variation is not accounted for. But from $\frac{dP}{dz} = -\rho g$, using Clausius-Clapeyron's equation

$P = nk_B T$ (k_B = Boltzmann's constant $= 1.38 \times 10^{-23}$ J/K $= \frac{R}{N}$

$= \dfrac{\text{universal gas constant}}{\text{Avogadro Number}} = \dfrac{8.314 \text{ J/mol K}}{6.023 \times 10^{23}}$), M = molecular weight of gas.

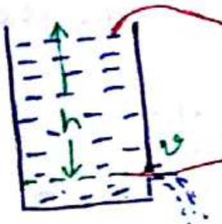
So $\dfrac{dP}{dz} = -\rho g = -\dfrac{Mg}{RT} P$ [As, $mn = \rho$, $mN = M$, $K_B = \dfrac{R}{N}$] [As $P = n k_B T = \dfrac{\rho}{m} \dfrac{R}{N} T = \dfrac{\rho R}{M} T$]

or $\dfrac{dP}{P} = -\dfrac{Mg}{RT} dz$

or $\ln P = -\dfrac{Mg}{RT} z + \ln P_0$

or $\boxed{P = P_0 e^{-Mgz/RT}}$ This is called "law of atmosphere."

Torricelli's theorem Velocity of efflux of a liquid through an orifice is equal to the velocity attained by a body in falling freely from the surface of the liquid to the orifice.



Total energy = KE + PE + pressure = $0 + gh + 0 = gh$.

Total energy = $\frac{1}{2}v^2 + 0 + 0$. or $\frac{1}{2}v^2 = gh$ or $v = \sqrt{2gh}$

Eulerian and Lagrangian description of conservation laws

The rate of change of a field variable $\phi(t, \bar{x})$ with respect to fixed position of space is called Eulerian derivative $\dfrac{\partial \phi}{\partial t}$ while derivative following a moving parcel is called Lagrangian derivative or substantial derivative or material derivative $\dfrac{D\phi}{Dt}$

$$\frac{D\phi}{Dt} = \underbrace{\frac{\partial \phi}{\partial t}}_{\text{local rate of change}} + \underbrace{\vec{V}\cdot\vec{\nabla}\phi}_{\text{convective rate of change}}$$

So changes in the properties of a moving fluid can be measured either on a fixed point in space while fluid particles are crossing it (Eulerian) or by following a fluid parcel along its path (Lagrangian)

Eulerian $v(t, \vec{x}(\vec{x}_0, t)) = \dfrac{\partial}{\partial t}\vec{X}(t, \vec{x}_0)$

Reynold's transport theorem As we have defined our conservation laws in Lagrangian description, Reynold's transport theorem gives the Eulerian equivalent of the integral taken over a moving material volume of a fluid.

$$\left(\frac{d\phi}{dt}\right)_{\substack{\text{material}\\ \text{volume}}} = \frac{d}{dt}\int_{V(t)} \psi\rho\, dV + \iint_{S(t)} \psi\rho\vec{v}\cdot\hat{n}\, ds \,, \quad \left[\psi = \frac{d\phi}{dm} = \text{amount of } \phi \text{ per unit mass}\right]$$

Divergence theorem

$$= \int_V \left[\frac{\partial}{\partial t}(\psi\rho) + \vec{\nabla}\cdot(\rho\vec{v}\psi)\right] dV = \int_V \left[\frac{D}{Dt}(\rho\psi) + \rho\psi\vec{\nabla}\cdot\vec{v}\right] dV$$

Notice that conservation law indicate no source or sink meaning $\cancel{\frac{d\phi}{dm}}$ $\frac{dm}{dt} = 0$, meaning $\psi = \frac{d\phi}{dm} = 1$ when $\phi = m$.

$$\boxed{\frac{D\rho}{Dt} + \rho\vec{\nabla}\cdot\vec{v} = 0}$$

mass conservation law in Eulerian coordinate system.

So incompressibility $(\vec{\nabla}\cdot\vec{v} = 0)$ means $\frac{D\rho}{Dt} = 0$. or ρ is not a constant but ρ does not change along a streamline.

In presence of external force $\vec{f}$ per unit volume, the non-conservative form with $\psi = \vec{v}$ is

$$\frac{D}{Dt}(\rho\vec{v}) + \rho\vec{v}\,\vec{\nabla}\cdot\vec{v} = \vec{f}$$

check!

$$\Rightarrow \rho\frac{D\vec{v}}{Dt} + v\underbrace{\left(\frac{D\rho}{Dt} + \rho\vec{\nabla}\cdot\vec{v}\right)}_{= 0 \text{ (continuity)}} = \vec{f} \Rightarrow \boxed{\rho\frac{D\vec{v}}{Dt} = \vec{f}}$$

Using Reynold's transport theorem, we find the conservative form

$$\frac{\partial}{\partial t}(\rho\vec{v}) + \vec{\nabla}\cdot(\underbrace{\rho\vec{v}\vec{v}}_{\text{Dyadic product}}) = \vec{f} = \vec{f}_{surface} + \vec{f}_{body}$$

Now $\int_V \vec{f}_s\, dV = \int_S \vec{\vec{\sigma}}\cdot\hat{n}\, ds = \int_V \vec{\nabla}\cdot\vec{\vec{\sigma}}\, dV$ where $\vec{\vec{\sigma}}$ = Stress tensor or stress matrix.

$$\vec{\vec{\sigma}} = -\begin{pmatrix} p & 0 & 0 \\ 0 & p & 0 \\ 0 & 0 & p \end{pmatrix} + \begin{pmatrix} \sigma_{xx}+p & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy}+p & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz}+p \end{pmatrix}$$

$= -p\bar{\bar{\mathbb{I}}} + \bar{\bar{\tau}}$ ↳ Deviatoric / viscous stress tensor.

↳ Thermodynamic pressure ($= nk_BT$) . So $\vec{\nabla}\cdot\bar{\bar{\sigma}} = -\vec{\nabla}p + \vec{\nabla}\cdot\bar{\bar{\tau}}$.

$\vec{f}_b = \rho\vec{g} - 2\rho\vec{\omega}\times\vec{v} - \rho\vec{\omega}\times(\vec{\omega}\times\vec{r})$. As gravitation & centrifugal forces

(gravity, coriolis force, centrifugal force)

are dependent on position but not velocity, so they can be absorbed into a modified pressure & hence effectively ignored. Coriolis force however has to be treated explicitly.

So conservation of momentum equation becomes,

$$\boxed{\frac{\partial}{\partial t}(\rho\vec{v}) + \vec{\nabla}\cdot(\rho\vec{v}\vec{v}) = -\vec{\nabla}p + \vec{\nabla}\cdot\vec{\tau} + \vec{f}_b}$$

Stress tensor for Newtonian fluid

$$\vec{\vec{\tau}} = \eta\{\vec{\nabla}\vec{v} + (\vec{\nabla}\vec{v})^T\} + \lambda(\vec{\nabla}\cdot\vec{v})\mathbb{I} \qquad [\lambda = -\tfrac{2}{3}\mu]$$

η ↳ viscosity coefficient (molecular); λ ↳ viscosity coefficient (bulk)

So $$\frac{\partial}{\partial t}(\rho\vec{v}) + \vec{\nabla}\cdot(\rho\vec{v}\vec{v}) = -\vec{\nabla}p + \vec{\nabla}\cdot[\eta\{\vec{\nabla}\vec{v} + (\vec{\nabla}\vec{v})^T\}] + \vec{\nabla}(\lambda\cancel{\vec{\nabla}\cdot\vec{v}}^{\,0}) + \vec{f}_b$$

incompressible flow

$$= -\vec{\nabla}p + \cancel{\eta\nabla^2\vec{v}}^{\,0} + \vec{f}_b = -\vec{\nabla}p + \vec{f}_b$$

inviscid flow

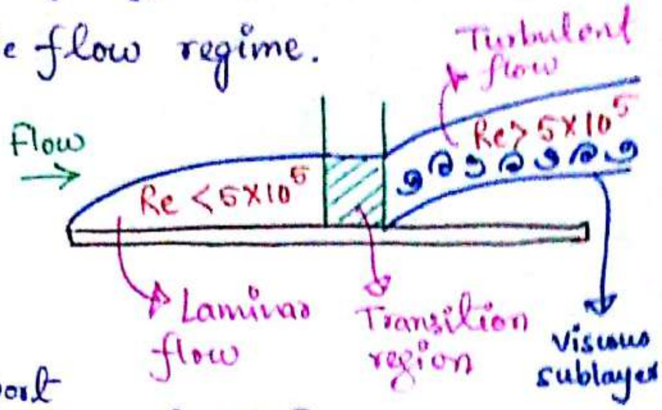
Similarly energy conservation equation can be derived.

General form $$\frac{\partial}{\partial t}(\rho\phi) + \vec{\nabla}\cdot(\rho\vec{v}\phi) = \vec{\nabla}\cdot(\Gamma\vec{\nabla}\phi) + S$$

[unsteady term] [convection term] [diffusion term] [source term]

Like Reynolds number $Re = \dfrac{\rho v L}{\eta} = \dfrac{\text{advection (inertia)}}{\text{diffusion (viscous)}}$

reveals the boundary layer characteristic of the flow if momentum fluxes are in same direction, e.g. if diffusion is in the cross-stream direction then Re conveys the flow regime.



Similarly Schmidt number

$Sc = \frac{\text{momentum diffusivity}}{\text{mass diffusivity}}$

Peclet number $Pe = \frac{\text{advective transport}}{\text{diffusive transport}} = Re \times Sc$

where Pr = prandtl number $= \frac{\text{hydrodynamic boundary layer}}{\text{thermal boundary layer}}$

$= \frac{\text{convection}}{\text{conduction}}$.

Grashof number $= \frac{\text{buoyancy force}}{\text{Viscous force}}$

Mach number (M) $= \frac{\text{object speed}}{\text{speed of sound}}$ where $v_{sound} = \sqrt{\gamma \left(\frac{\partial p}{\partial \rho}\right)_T}$

$= \sqrt{\gamma R T}$ (ideal gas)

$\gamma = c_p / c_v$

$M < 0.2$ is incompressible flow.
$M < 1$ is subsonic flow.
$M = 1$ is sonic flow.
$M < 5$ is supersonic flow
$M > 5$ is hypersonic flow.

flow classification

Euler equation in Lagrangian form

$$\rho\left(\frac{\partial \vec{v}}{\partial t} + (\vec{v}.\vec{\nabla}) v\right) = -\vec{\nabla} p - \rho \vec{\nabla} \varphi + \eta \nabla^2 \vec{v} \quad \& \quad \vec{\Omega} = \vec{\nabla} \times \vec{v}$$

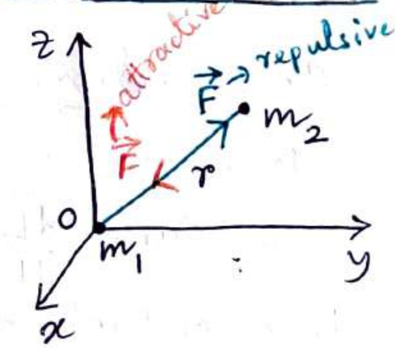
gives, $\frac{\partial \vec{\Omega}}{\partial t} + \vec{\nabla} \times (\vec{\Omega} \times \vec{v}) = \frac{\eta}{\rho} \nabla^2 \vec{\Omega}$

↳ specific viscosity $= \frac{1}{Re}$

If we non-dimensionalize the equation (see Feynman lectures, vol 2 section 41.3), then $\frac{\partial \vec{\Omega}}{\partial t} + \vec{\nabla} \times (\vec{\Omega} \times \vec{v}) = \frac{1}{Re} \nabla^2 \vec{\Omega}$

So if Mach number & Reynold's number are same then velocities near or above sound speed in two scenarios are equivalent.

Central Force & Planetary Motion (Inverse-square law Forces)



Suppose there are two masses m_1 & m_2 are enacted by a force $\vec{F}$ whose magnitude depends only on the distance between two masses and is always directed along the line joining the masses. Such force is termed as central force $\vec{F} = f(r)\hat{r} = f(r)\frac{\vec{r}}{|\vec{r}|}$. When $f(r) < 0$, the force is attractive and when $f(r) > 0$, the force is repulsive.

Examples: (A) Gravitational force of attraction:

The gravitational force of attraction between two isolated point masses m_1 and m_2 is $\vec{F} = -\frac{G m_1 m_2}{r^2}\hat{r}$. $G = 6.67 \times 10^{-11}\ m^3/kg\text{-}s^2$ and $f(r)$ is an inverse-square law force.

(B) Electrostatic force of attraction/repulsion:

The electrostatic force of attraction between two isolated point charges $+q_1$ and $-q_2$ in vacuum is $\vec{F} = -\frac{q_1 q_2}{4\pi\epsilon_0 r^2}\hat{r}$. Here the Coulomb's constant $= \frac{1}{4\pi\epsilon_0} \approx 9\times 10^9\ Nm^2/C^2$ and $f(r)$ is also an inverse-square low force. The force becomes repulsive if q_2 changes sign. Non inverse-square law forces, e.g. the motion of a particle in the "spiral of Archimedes" ($r = K\phi$) are central force as well displaying $r^{-5} + r^{-3}$ nature.

Nature of Central force: Due to the inherent symmetry of $f(r)$ which is not a function of θ & ϕ, for a given r the same magnitude of force acts along any direction of the position vector $\vec{r}$, or $f(r)$ is spherically symmetric.

Now $\vec{\nabla}\times\vec{F} = \vec{\nabla}\times f(r)\hat{r} = f(r)\,\vec{\nabla}\times\hat{r} = \vec{0}$. So central forces are conservative & canbe expressed as gradient of scalar as $\vec{\nabla} \times \vec{\nabla} V = 0$ for any scalar point function V.

Conservation of Angular momentum :

We know that torque $\vec{\tau}$ is the rate of change of angular momentum $\frac{d\vec{L}}{dt} = \dot{\vec{L}}$. Torque experienced on a particle subjected to a central force is $\vec{\tau} = \vec{r} \times \vec{F} = \vec{r} \times f(r)\hat{r} = \vec{0}$.

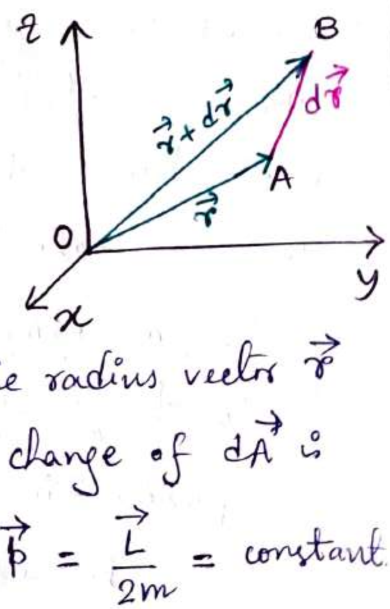
Therefore $\frac{d\vec{L}}{dt} = \vec{0}$ or $\vec{L}$ is a constant vector. and called as "constant of motion" (COM).

Trajectory of particle (orbit):

As $\vec{\tau} = \vec{0}$ yields $\vec{L} = \vec{r} \times \vec{p}$ a constant vector ($\vec{p}$ is linear momentum), we have $\vec{r} \cdot \vec{L} = \vec{r} \cdot \vec{r} \times \vec{p} = \vec{r} \times \vec{r} \cdot \vec{p} = 0$

Therefore $\vec{r}$ is always $\perp^r$ to $\vec{L}$ and the motion is always confined to a plane containing $\vec{r}$ and $\vec{p}$.

Areal velocity :



Let $m = \frac{m_1 m_2}{m_1 + m_2}$ is the reduced mass of the particle and the instantaneous position vector at A ($\vec{r}$) changes to B ($\vec{r} + d\vec{r}$), so that $\vec{AB} = d\vec{r}$. The area $d\vec{A}$ = OAB swept by the radius vector $\vec{r}$ in time dt is $d\vec{A} = \frac{1}{2}\vec{r} \times d\vec{r}$. The rate of change of $d\vec{A}$ is

$$\text{Areal velocity} = \frac{d\vec{A}}{dt} = \frac{1}{2}\vec{r} \times \frac{d\vec{r}}{dt} = \frac{1}{2m}\vec{r} \times \vec{p} = \frac{\vec{L}}{2m} = \text{constant}$$

Thus the radius vector sweeps area at a constant rate and areal velocity under central force is a "COM."

Plane of Earth's orbit is called an Ecliptic.

Equations of motion under Central force :

Let us consider the confined planar motion of the particle in xy plane & the coordinates at any instant is given in spherical polar coordinates (r, θ).

We know acceleration $\vec{a} = (\underbrace{\ddot{r} - r\dot{\theta}^2}_{\text{radial}})\hat{r} + (\underbrace{r\ddot{\theta} + 2\dot{r}\dot{\theta}}_{\text{transverse}})\hat{\theta}$

So mass x acceleration = net force we have

$m(\ddot{r} - r\dot{\theta}^2)\hat{r} + m(r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{\theta} = f(r)\hat{r}$. Equating we have

$m(\ddot{r} - r\dot{\theta}^2) = f(r)$, $m(r\ddot{\theta} + 2\dot{r}\dot{\theta}) = 0$. These are the radial & transverse forces. To note from the later,

$$m(r\ddot{\theta} + 2\dot{r}\dot{\theta}) = 0 = \frac{m}{r}(r^2\ddot{\theta} + 2r\dot{r}\dot{\theta}) = \frac{m}{r}\frac{d}{dt}(r^2\dot{\theta})$$

$\therefore \frac{d}{dt}(r^2\dot{\theta}) = 0$ gives $r^2\dot{\theta} = h =$ constant.

We know $|\vec{L}| = |\vec{r} \times \vec{p}| = |\vec{r} \times m\vec{v}| = m|\vec{r} \times (\dot{r}\hat{r} + r\dot{\theta}\hat{\theta})|$

$= mr^2\dot{\theta} = mh =$ constant, so we get back our known result.

Law of conservation of Energy (COM):

According to the radial equation of motion $f(r) = m(\ddot{r} - r\dot{\theta}^2)$

from $L = mr^2\dot{\theta}$ we have $\dot{\theta} = L/mr^2$. Substituting we get

$f(r) = m\ddot{r} - \frac{L^2}{mr^3}$. Due to conservative nature of central force we know $f(r) = -\frac{dV}{dr}$ where V is the scalar potential.

$$\therefore m\ddot{r} - \frac{L^2}{mr^3} = -\frac{dU}{dr} \quad \text{or} \quad m\ddot{r} = -\frac{d}{dr}\left[V + \frac{L^2}{2mr^2}\right]$$

Multiplying both sides with $\dot{r}$ we have

$$m\dot{r}\ddot{r} = +\dot{r}\frac{d}{dr}\left[V + \frac{L^2}{2mr^2}\right] \quad \text{or} \quad \frac{d}{dt}\left(\frac{1}{2}m\dot{r}^2\right) = -\frac{d}{dt}\left[V + \frac{L^2}{2mr^2}\right]$$

$$\text{or} \quad \frac{d}{dt}\left[\frac{1}{2}m\dot{r}^2 + \frac{L^2}{2mr^2} + V\right] = 0 \quad \text{or} \quad \underbrace{\frac{1}{2}m\dot{r}^2}_{\text{[translational KE]}} + \underbrace{\frac{L^2}{2mr^2}}_{\text{[energy due to centripetal force]}} + \underbrace{U}_{\text{[potential energy]}} = \text{constant}$$

$\therefore \; \frac{1}{2}m\dot{r}^2 + \frac{L^2}{2mr^2} + V = E$ where E is the total energy.

Hence E is a COM in central force system.

<u>Radial Equation in other variable:</u>

By substituting $r = \frac{1}{u}$ in $m(\ddot{r} - r\dot{\theta}^2) = f(r)$ and recalibrating $r^2\dot{\theta} = h$ so that $\dot{\theta} = \frac{h}{r^2} = hu^2$ and $\ddot{r} = \frac{d}{dt}(\dot{r}) = \frac{d}{dt}\left(\frac{dr}{d\theta}\frac{d\theta}{dt}\right)$

$$= \frac{d}{dt}\left(\frac{dr}{d\theta}\frac{h}{r^2}\right) = \frac{d}{dt}\left(-\frac{1}{\cancel{u^2}}\frac{du}{d\theta}h\cancel{u^2}\right) = \frac{d}{dt}\left(-h\frac{du}{d\theta}\right) = \frac{d}{d\theta}\left(-h\frac{du}{d\theta}\right)\frac{d\theta}{dt}$$

$$= -h\frac{d^2u}{d\theta^2}hu^2 = -h^2u^2\frac{d^2u}{d\theta^2}.$$ So the radial force eq^n becomes

$$m\left(-h^2u^2\frac{d^2u}{d\theta^2} - \frac{1}{u}h^2u^4\right) = f(1/u) \qquad [\because L = mh]$$

or $$\boxed{\frac{d^2u}{d\theta^2} + u = -\frac{f(1/u)}{mh^2u^2}.} = -\frac{mf(1/u)}{L^2u^2}$$

We can rewrite the Energy conservation in terms of u; noting $L = mr^2\dot{\theta}$ we have $\frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\dot{\theta}^2 + V = E$

or $\frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) - \int f(r)\,dr = E.$

Substituting $\dot{\theta} = hu^2$ and $\dot{r} = -h\frac{du}{d\theta}$ in $v^2 = \dot{r}^2 + r^2\dot{\theta}^2$ we have

$v^2 = h^2\left(\frac{du}{d\theta}\right)^2 + \frac{1}{u^2}h^2u^4 = h^2\left[\left(\frac{du}{d\theta}\right)^2 + u^2\right].$ So the COM eq^n becomes

$$\frac{1}{2}mh^2\left[\left(\frac{du}{d\theta}\right)^2 + u^2\right] + V = E \quad \text{or} \quad \boxed{\left(\frac{du}{d\theta}\right)^2 + u^2 = \frac{2(E-V)}{mh^2}.} = \frac{2m(E-V)}{L^2}$$

We can back substitute $r = 1/u$ to show

$$\boxed{\left(\frac{dr}{d\theta}\right)^2 + r^2 = \frac{2(E-V)r^4}{mh^2}.}$$

<u>Solution of radial eq^n for attractive inverse-square force:</u>

The generic form of such force field is $f(r) = -K/r^2$, $K > 0$.

In case of Newton's law of Gravitation, $K = Gm_1m_2$. From $f(r) = -\frac{\partial V}{\partial r} = -\frac{K}{r^2}$ we have by integrating $V = -\frac{K}{r}$ assuming $V(r) = 0$ at $r = \infty$.

or $f(1/u) = -Ku^2.$

Substituting in radial eqn. we get $\frac{d^2u}{d\theta^2} + u = \frac{m}{L^2u^2} Ku^2$

or $\frac{d^2u}{d\theta^2} + u - \frac{mK}{L^2} = 0$. Let $x = u - \frac{mK}{L^2}$, $\frac{d^2u}{d\theta^2} = \frac{d^2x}{d\theta^2}$

or $\frac{d^2x}{d\theta^2} + x = 0$ which is the linear 2nd order differential equation of SHM whose solution is $x = A\cos\theta$ by choosing zero phase.

$\therefore\ u - \frac{mK}{L^2} = A\cos\theta$ or $\frac{1}{r} = \frac{mK}{L^2}\left(1 + \frac{L^2A}{mK}\cos\theta\right)$

or $\frac{1}{r} = \frac{1}{l}(1 + \epsilon\cos\theta)$ where $l = \frac{L^2}{mK}$, $\epsilon = Al$ = eccentricity

which is the equation of a conic section in (r,θ) coordinate.

$\therefore$ Depending on ϵ (in other words L and A) the shape of the trajectory of particle is (a) Hyperbola for $\epsilon > 1$, (b) parabola for $\epsilon = 1$, (c) ellipse for $0 < \epsilon < 1$ and (d) circle for $\epsilon = 0$.

Eccentricity in terms of Energy:

Recall the radial eqn. in terms of Energy i.e. $\left(\frac{du}{d\theta}\right)^2 + u^2 = \frac{2m(E-V)}{L^2}$

Now $V = -\frac{K}{r} = -Ku$ and from $u = \frac{mK}{L^2} + A\cos\theta$, we have

$\frac{du}{d\theta} = -A\sin\theta$. Substituting,

$$(A\sin\theta)^2 + \left(\frac{mK}{L^2} + A\cos\theta\right)^2 = \frac{2mE}{L^2} + \frac{2mK}{L^2}\left(\frac{mK}{L^2} + A\cos\theta\right)$$

$$\text{or } A^2 + \frac{2mKA}{L^2}\cancel{\cos\theta} + \frac{m^2K^2}{L^4} = \frac{2mE}{L^2} + \frac{2m^2K^2}{L^4} + \frac{2mKA}{L^2}\cancel{\cos\theta}$$

$$\text{or } A^2 = \frac{m^2K^2}{L^4} + \frac{2mE}{L^2} \quad \text{or} \quad A = \sqrt{\frac{m^2K^2}{L^4} + \frac{2mE}{L^2}}$$

$$\therefore\ \epsilon = Al = \frac{L^2}{mK}\sqrt{\frac{m^2K^2}{L^4} + \frac{2mE}{L^2}} = \sqrt{1 + \frac{2L^2E}{mK^2}}.$$

So the orbit equation becomes $\boxed{\frac{1}{r} = \frac{mK}{L^2}\left[1 + \sqrt{1 + \frac{2L^2E}{mK^2}}\cos\theta\right]}$

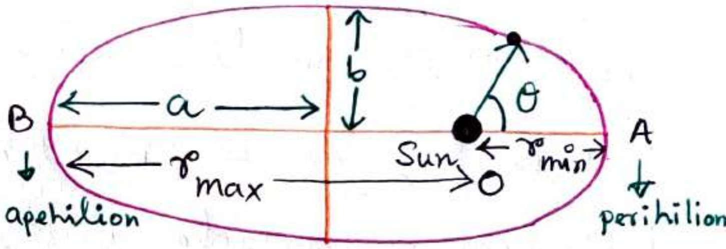
We note that when $\theta = 0$ or 2π, $\cos\theta = 1$ or $\frac{1}{r}$ has maximum value

$\frac{1}{r} = \frac{mK}{L^2} + A = \frac{mK}{L^2}\left[1 + \sqrt{1 + \frac{2L^2E}{mK^2}}\right]$. Similarly for $\theta = \pi$, $\cos\theta = -1$

$\frac{1}{r} = \frac{mK}{L^2} - A = \frac{mK}{L^2}\left[1 - \sqrt{1 + \frac{2L^2E}{mK^2}}\right]$ has minimum value.

Turning Points: We know that total energy is conserved (COM), so $E = \frac{1}{2}m\dot{r}^2 + \frac{L^2}{2mr^2} + V = \frac{1}{2}m\dot{r}^2 + \frac{L^2}{2mr^2} - \frac{K}{r}$.
Turning points are where r is maximum or minimum, i.e. the radial velocity $\dot{r} = 0$, meaning the translational kinetic energy is zero, so $E = \frac{L^2}{2mr^2} - \frac{K}{r}$ = Effective potential energy.
Also to note for repulsive force $E = \frac{1}{2}m\dot{r}^2 + \frac{L^2}{2mr^2} + \frac{K}{r} > 0$ for $K > 0$.
So $\epsilon = \sqrt{1 + \frac{2L^2E}{mK^2}} > 0$ or the trajectory is always a hyperbola.

Kepler's 1st Law of Elliptical Orbits: The law states every planet moves in an elliptical orbit around the Sun having Sun at one of the foci. The point at which the planet is closest to Sun is called perihilion



where $r_{min} = \frac{L^2}{mK(1+\epsilon)}$ and the furthest to Sun is called apehilion $r_{max} = \frac{L^2}{mK(1-\epsilon)}$ with $\epsilon < 1$ displaying Elliptic orbit having total energy negative $E < 0$. The semi-major axis "a" of the ellipse $= \frac{1}{2} \times$ sum of two apsidal distances (turning points)

$$= \frac{1}{2}(BO + OA) = \frac{1}{2}(r_{max} + r_{min}) = \frac{L^2}{2mK}\left[\frac{1}{1+\epsilon} + \frac{1}{1-\epsilon}\right]$$

$$= \frac{L^2}{mK(1-\epsilon^2)} = \frac{L^2/mK}{\cancel{1} - (\cancel{1} + \frac{2L^2E}{mK^2})} = -\frac{K}{2E} \quad \therefore \boxed{a = -\frac{K}{2E} \quad \text{or} \quad E = -\frac{K}{2a}}$$

So total energy depends only on semi-major axis a and is independent of L.
The semi-minor axis of the ellipse "b" is $b = a\sqrt{1-\epsilon^2}$

$$= a\sqrt{-\frac{2L^2E}{mK^2}} = a\sqrt{-\frac{2E}{K}}\sqrt{\frac{L^2}{mK}} = \sqrt{a}\frac{L}{\sqrt{mK}} = L\sqrt{\frac{a}{mK}}.$$

So b not only depends on Energy E, but on angular momentum L as well.

as $a(1-\epsilon^2) = \frac{L^2}{mK}$, the equation of ellipse is $\frac{1}{r} = \frac{1+\epsilon\cos\theta}{a(1-\epsilon^2)}$

Kepler's 2^{nd} Law of areas : The radius vector drawn from Sun to a planet sweeps out equal areas in equal times i.e. the areal velocity of radius vector is constant. $\frac{d\vec{A}}{dt} = \frac{\vec{L}}{2m} =$ constant

Kepler's 3^{rd} law : The square of period of revolution of planet around the Sun is proportional to cube of semi-major axis of ellipse. So if T is the time-period of revolution then

$$A = \int_0^A dA = \int_0^T \frac{L}{2m}\,dt = \frac{LT}{2m}.$$ Since for ellipse $A = \pi ab$,

we have $\pi ab = \frac{LT}{2m} \Rightarrow T = \frac{2\pi ab m}{L} = \frac{2\pi a m}{\cancel{L}}\cancel{L}\sqrt{\frac{a}{mK}}$

or $T = 2\pi a^{3/2}\sqrt{\frac{m}{K}}$. Squaring both sides, $T^2 = \frac{4\pi^2 m}{K}a^3$

or $\boxed{T^2 \propto a^3}$. Also $m = \frac{m_1 m_2}{m_1+m_2}$ and $K = G m_1 m_2$. $T^2 = \frac{4\pi^2 a^3}{G(m_1+m_2)}$

for planetary motion. Similarly for attractive force between electron of charge $-e$ and nucleus of charge Ze, $K = Ze^2/4\pi\epsilon_0$ so the orbit is elliptical and $T^2 = \frac{4\pi^2 a^3 m}{Ze^2}4\pi\epsilon_0$

$m = \frac{m_e m_N}{m_e+m_N}$,

or $T = \frac{4\pi}{e}a^{3/2}\sqrt{\frac{\pi\epsilon_0 m}{Z}}$.

CW In plane polar coordinates the equation of the spiral of Archimedes is $r = K\theta$. If a particle of mass m moves along this curve, determine the r dependence of the central force.

$r = K\theta$ or $u = \frac{1}{r} = \frac{1}{K\theta}$. Equation $\Rightarrow \frac{d^2u}{d\theta^2} + u = -\frac{mf(1/u)}{L^2u^2}$

$\frac{d^2u}{d\theta^2} = \frac{d}{d\theta}\left(-\frac{1}{K\theta^2}\right) = \frac{2}{K\theta^3}$, so $\frac{2}{K\theta^3} + \frac{1}{K\theta} = -\frac{m}{L^2u^2}f(1/u)$

or $f(1/u) = -\frac{L^2u^2}{m}\left(\frac{2}{K\theta^3}+\frac{1}{K\theta}\right) = -\frac{L^2}{m}\frac{1}{K^2\theta^2}\left(\frac{2}{K\theta^3}+\frac{1}{K\theta}\right) = -\frac{L^2}{m}\left(\frac{2}{K^3\theta^5}+\frac{1}{K^3\theta^3}\right)$

or $f(r) = -\frac{L^2}{m}\left(\frac{2K^2}{K^5\theta^5}+\frac{1}{K^3\theta^3}\right) = -\frac{L^2}{m}\left(\frac{2K^2}{r^5}+\frac{1}{r^3}\right)$.

CW The diameter of the Neptune's orbit around the Sun is 30 times the diameter of the Earth's orbit around the Sun. Calculate the period of revolution of Neptune.

Using Kepler's 3rd law, $\frac{T_{Neptune}^2}{T_{Earth}^2} = \frac{a_{Neptune}^3}{a_{Earth}^3}$. Given $\frac{a_{Neptune}}{a_{Earth}} \approx 30$,

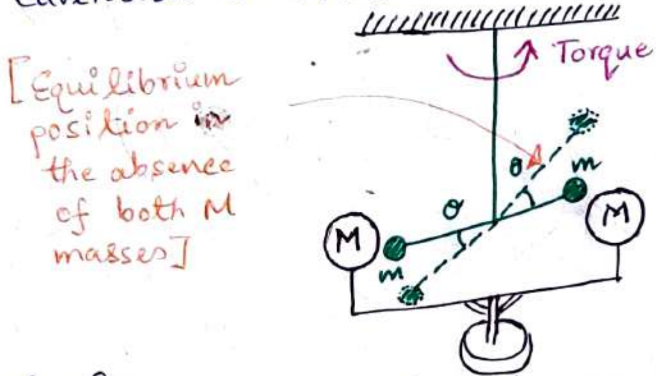
$T_{Earth} = 1$ year, so, $T_{Neptune} = \sqrt{30^3}$ year ≈ 164 years.

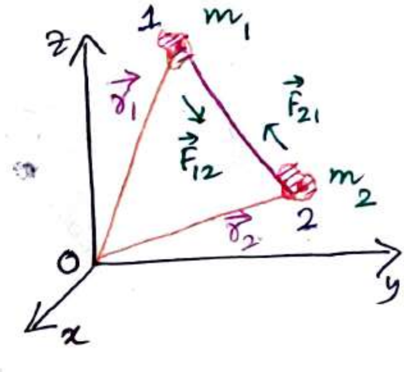
Gravitation

Nature of Gravitational Force: We know that there are four forces in nature: electromagnetic, strong & weak force (nuclear), and the weakest of the former, i.e. gravitational force. It doesn't have any significant effect at atomic/molecular levels. The law of grav. attraction was established by Newton in 1687 & experimentally demonstrated by Cavendish in 1797.

M = Large Lead spheres

m = small Lead spheres



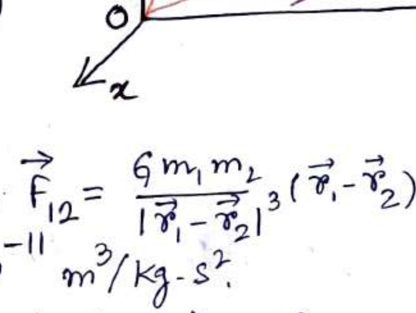


The Gravitational force on m_1 due to m_2: $\vec F_{12} = \frac{G m_1 m_2}{|\vec r_1 - \vec r_2|^3}(\vec r_1 - \vec r_2)$

where $G = 6.673 \times 10^{-11}\ Nm^2/kg^2 = 6.673 \times 10^{-11}\ m^3/kg\text{-}s^2$.

Newton's law is applicable only to bodies having dimensions negligible compared to the separating distance, otherwise $|\vec r_1 - \vec r_2|$ isn't precisely defined. For extended object we have, using Newton's 3rd law $\vec F_{21} = -\vec F_{12}$, $\delta \vec F_{ij} = \frac{G\,\delta m_i\,\delta m_j}{|\vec r_{ij}|^2}\vec r_{ij}$ and the net force is

$$\vec F = \sum_{i,j} \frac{G\,\delta m_i\,\delta m_j}{|\vec r_{ij}|^2}\vec r_{ij}.$$



Gravitational Field & Potentials

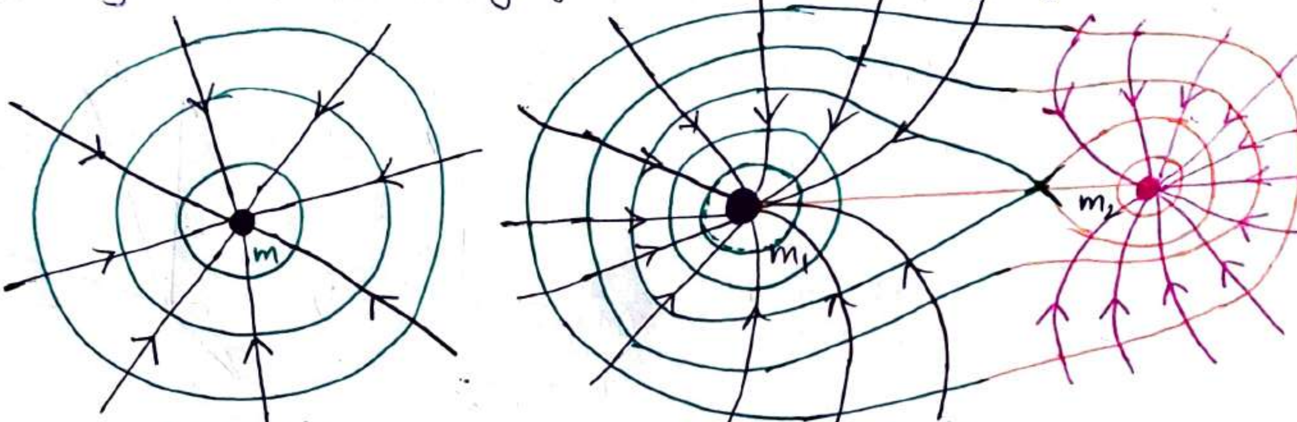
The Grav. force $\vec F_m$ acting on m at $\vec r$ due to extended masses δm_i at $\vec r_i$ is

$$\vec F_m = \sum_i \frac{G m\,\delta m_i}{|\vec r - \vec r_i|^3}(\vec r - \vec r_i) \underset{\text{continuum limit}}{\equiv} \int_V \frac{G m(\vec r - \vec r')}{|r - \vec r'|^3}\rho(\vec r')\,dV'$$

So force per unit mass acting on m is the gravitational field

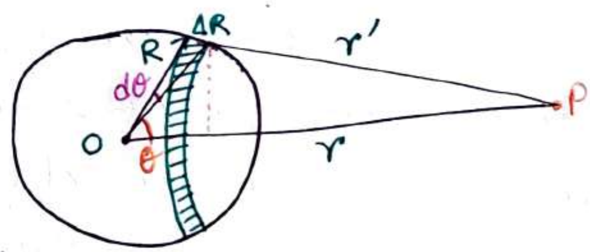
$$\vec g(\vec r) = \lim_{m\to 0}\frac{\vec F_m}{m} = G\int_V \frac{\vec r - \vec r'}{|\vec r - \vec r'|^3}\rho(\vec r')\,dV'.$$

Since Gravitational forces are central forces or conservative in nature, $\vec{g} = \vec{\nabla} V(\vec{r})$ where $V(\vec{r}) = -G\int \frac{\rho(\vec{r}')\,dV'}{|\vec{r}-\vec{r}'|}$ is the Gravitational potential. While V is a scalar & $\vec{g}$ is a vector, it is easier to handle $\vec{V}$ instead of $\vec{g}$. Moreover, just like electric fluxes as lines of force, we can do it for grav. force.



Monopole Dipole

Gravitational Potential of a thin spherical shell :



Let us consider a homogeneous shell of mass M, density ρ radius R & thickness ΔR. We want to calculate grav. potential at an inside & outside point P. Consider a ring of radius $R\sin\theta$ & width $\Delta R = R\,d\theta$ so that mass of the ring δM = volume $\times$ density $= (\Delta R)(2\pi R\sin\theta\,\Delta R)\rho = 2\pi R^2 \Delta R \sin\theta\, d\theta\, \rho$ $[\because \pi(R\sin\theta+\Delta R)^2 - \pi(R\sin\theta)^2 \simeq 2\pi R\sin\theta\,\Delta R]$

So Grav. pot. at P due to this mass is

$$dV = -\frac{G\, 2\pi R^2 \Delta R \sin\theta\, d\theta\, \rho}{r'}$$

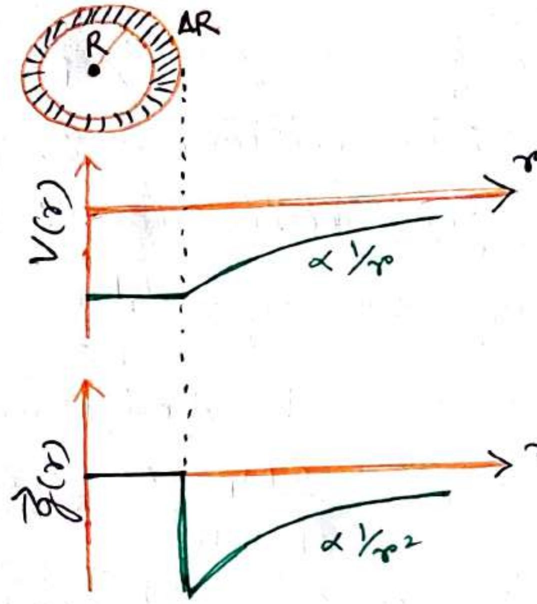
Now from triangle law of cosines, $r'^2 = R^2 + r^2 - 2Rr\cos\theta$

Differentiating we have $\cancel{2}r'dr' = \cancel{2}Rr\sin\theta\,d\theta$.

So total potential at P due to the spherical shell is

$$V = -\int \frac{2\pi R^{\cancel{2}}\Delta R \rho G}{\cancel{r'}}\,\frac{\cancel{r'}dr'}{\cancel{R}r} = -\frac{2\pi R \Delta R \rho G}{r}\int dr'$$

If P lies outside the shell then $V = -\frac{2\pi R \Delta R \rho G}{r}\int_{r-R}^{r+R} dr'$ $= -G\rho\left[\frac{4\pi R^2 \Delta R}{r}\right] = \boxed{-\frac{GM}{r} \text{ for } r > R}$ and the grav. field is $\vec{g} = \vec{\nabla} V(r) = \boxed{\frac{GM}{r^2}\hat{r}.}$ If P lies inside the shell then $V = -\frac{2\pi R \Delta R \rho G}{r}\int_{R-r}^{R+r} dr' = -4\pi R \Delta R \rho G = -G\rho\frac{4\pi R^2 \Delta R}{R} = \boxed{-\frac{GM}{R}}$ and the field $\vec{g} = \vec{\nabla} V(r) = \boxed{\vec{0}. \text{ for } r < R.}$



Gravitational Potential of a sphere of uniform density:

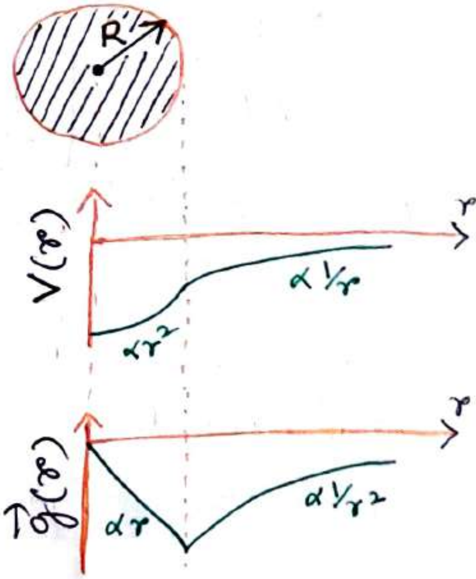
Consider a sphere of radius R, density ρ and mass M. Grav. potential at P at a distance r from the center inside the sphere will have two contributions - one due to the small sphere of radius r for which P is an external point and the other due to ~~spherical~~ spherical shell of thickness $R-r$ for which P is internal point. If M' is mass of enclosed sphere of radius r then $\frac{M'}{M} = \frac{r^3}{R^3}$ so that $V(r) = -\frac{GM'}{r} = -\frac{GMr^2}{R^3}$. To calculate the other part, consider a shell of radius r' $(r < r' < R)$ and thickness dr'. Gravitational potential due to this shell is $V = -\int \frac{G\rho\, 4\pi r'^2 dr'}{r'}$

$$= -4\pi G\rho\int_r^R r' dr' = -2\pi G\rho(R^2 - r^2) = -G\left\{\left(\frac{4\pi}{3}\rho R^3\right)\left(\frac{3}{2R^3}\right)\right\}(R^2 - r^2)$$
$$= -GM\left(\frac{3}{2R^3}\right)(R^2 - r^2).$$

$\therefore$ Net Grav. pot. at P $= -\frac{GMr^2}{R^3} - \frac{3GM}{2R^3}(R^2 - r^2) = \boxed{-\frac{3GM}{2R} + \frac{GMr^2}{2R^3}}$

Grav. field $\vec{g}(r) = \vec{\nabla} V(r) = \boxed{\frac{GM}{R^3}\vec{r}}$ for $r < R$. For $r > R$ point P is always an external point, so using previous calculation of spherical shell, we have

$$\boxed{V(r) = -\frac{GM}{r} \text{ for } r > R}\ ; \quad \boxed{\vec{g}(\vec{r}) = \vec{\nabla} V(r) = \frac{GM}{r^2}\hat{r}}$$



Gravitational Acceleration on the Earth's surface: The grav. force at the surface of Earth on a (gravitational) mass m_G is $\vec{F} = \frac{GM_E m_G}{R_E^2}\hat{r}$. From Newton's law we have, $\vec{F} = m_I \vec{a}$ where m_I is the (inertial) mass. When $m_I = m_G$ then grav. acceleration is independent of m_G and $g_0 = \frac{GM_E}{R_E^2} = 9.8\ m/s^2$ (at Equator). The grav. acceleration at a distance h **above** Earth's surface is ($R_E >> h$)

$$\vec{g}(r) = \frac{GM_E}{(R_E+h)^2}\hat{r} = \frac{GM_E}{R_E^2}\left[1+\left(\frac{h}{R_E}\right)\right]^{-2} \simeq \frac{GM_E}{R_E^2}\left(1-\frac{2h}{R_E}+\cdots\right)$$

or $\vec{g}(r) \simeq g_0\left(1-\frac{2h}{R_E}+\cdots\right)$.

For $h = 1\,km$ and $R_E \simeq 10^4\,km$, we have $\frac{2h}{R_E} \simeq 10^{-4}$.

$R_E = 6.637\times10^6\,m$, Earth's mean density $\rho = 5.57\times10^3\,kg/m^3$ & $G = 6.66\times10^{-11}\,Nm^2/Kg^2$, Earth's grav. potential at surface is

$$V = \frac{GM_E}{R_E} = \frac{G\frac{4}{3}\pi R_E^3\rho}{R_E} = \frac{4\pi\rho G R_E^2}{3} = \frac{4\pi\times5.57\times10^{13}\times6.66\times10^{-11}\times(6.637\times10^{6})^2}{3}$$

$= 6.845\times10^7\ J/kg$.

The grav. acceleration at a distance h **below** Earth's surface is

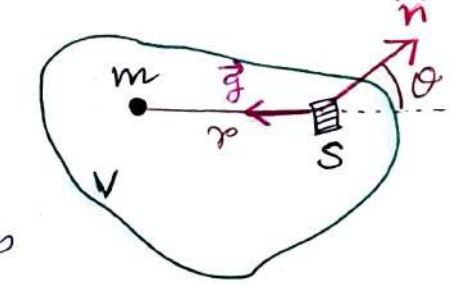
$\vec{g}(r) = \frac{GM_E'}{(R_E-h)^2}$ where M_E' is the Earth's mass at depth h =

$$M_E' = \frac{4}{3}\pi(R_E-h)^3\rho. \quad \therefore\ \vec{g}(r) = \frac{G\frac{4}{3}\pi\rho(R_E-h)^3}{(R_E-h)^2} = G\frac{4}{3}\pi\rho R_E\left(1-\frac{h}{R_E}\right)$$

$$= \frac{G\frac{4}{3}\pi\rho R_E^3}{R_E^2}\left(1-\frac{h}{R_E}\right) = \frac{GM_E}{R_E^2}\left(1-\frac{h}{R_E}\right) = g_0\left(1-\frac{h}{R_E}\right).$$ At $h = R_E$, $\vec{g} = 0$.

Gravitational Field Equations

More intriguing is to find the differential equation of $\vec{g}(\vec{r})$ and $V(r)$. While $\vec{g}(\vec{r}) = \vec{\nabla} V(r)$, we know $\vec{\nabla} \times \vec{g} = \vec{0}$ but this cannot uniquely determine $\vec{g}$ as the information of the distribution of matter is missing.



Consider a mass m contained in volume V and let us study the grav. field $\vec{g}(\vec{r})$ due to m. $\hat{n}$ = unit normal to surface S that bounds V and we calculate the flux of $\vec{g}$ through S

$$I = \oiint_S \vec{g}\cdot\hat{n}\, ds = -\oiint_S \frac{Gm}{r^2}\cos\theta\, ds = \oiint_S Gm\, d\Omega = -4\pi mG$$

where $d\Omega = \frac{ds\cos\theta}{r^2}$ is the solid angle subtended by $d\vec{S}$ at m. and $\int d\Omega = 4\pi$. Identically in electrostatics, there are 4π lines of force emanating from a unit charge. For a set of mass m_i we have

$$I = \oiint_S \vec{g}\cdot\hat{n}\, ds = -\sum_i 4\pi m_i G \underset{[\text{continuum}]}{=} -\iiint_V 4\pi G\rho\, dV$$

From Gauss's Divergence theorem, $\oiint_S \vec{g}\cdot\hat{n}\, ds = \iiint_V \vec{\nabla}\cdot\vec{g}\, dV$

$$\therefore \iiint_V (\vec{\nabla}\cdot\vec{g} + 4\pi G\rho)\, dV = 0$$

While this relation is true for arbitrary V, $\boxed{\vec{\nabla}\cdot\vec{g} + 4\pi\rho G = 0}$

→ Poisson's Equation

or $\boxed{\nabla^2 V + 4\pi\rho G = 0}$. This with the B.C. $r\to\infty$, $V\to 0$, $\vec{g}\to\vec{0}$ uniquely determines $\vec{g}$ and V. ρ is the electric charge density for electrostatic potential & for $\rho = 0$, we recover Laplace equation

$$\boxed{\nabla^2 V = 0}$$