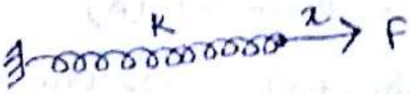
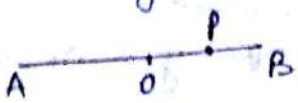
SHM Motion : Translation, rotation, vibration/oscillation

periodic motion $f(t) = f(t+T)$ e.g. $\sin\frac{2\pi t}{T}$, $\cos\frac{2\pi t}{T}$

if periodic over same path → oscillatory motion

elasticity & inertia





SHM is a special type of periodic motion where restoring force on particle is proportional to displacement & directed to the mean position.

oscillation between point A & B, mean position O. at time t, particle is at P & displacement is x. F - restoring force

$F \propto -x$ or $F = -kx$ or $ma = -kx$ "small oscillation approximation"

$\therefore \quad a = -\frac{k}{m}x = -\omega^2 x$

Characteristics (1) Linear motion → to-n-fro in straight line.
(2) $F \propto -x$

Linear harmonic motion ↔ angular harmonic motion.
(pendulum) (torsional pendulum)
$F \propto -x$ $\quad \tau \propto -\theta$

complete oscillation : one point to same point. (time period)

amplitude : maximum displacement on both sides

frequency : no. of oscillations in 1 second.

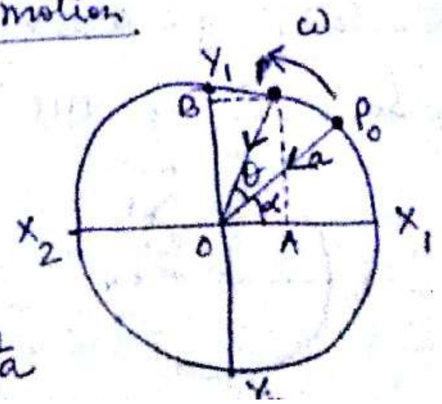
phase : displacement, velocity, accelaration & direction of motion. After 1 oscillation, phase is same.

t = 0, initial phase.

Relation between SHM & uniform circular motion

$\theta = \omega t$
$s = a\theta$

$OA = x$, $OB = y$
$= OP\cos(\theta+\alpha) = a\cos(\theta+\alpha)$
$= a\cos(\omega t + \alpha)$



speed $v = \omega a$, Centripetal acc $f_r = \frac{v^2}{a} = \omega^2 a$

Accelaration of A is component of f_r along X_1OX_2

$$f_A = -f_r \cos(\omega t+\alpha) = -\omega^2 a \cos(\omega t+\alpha) = -\omega^2 x$$

$$\therefore \underline{f_A \propto -x}.$$

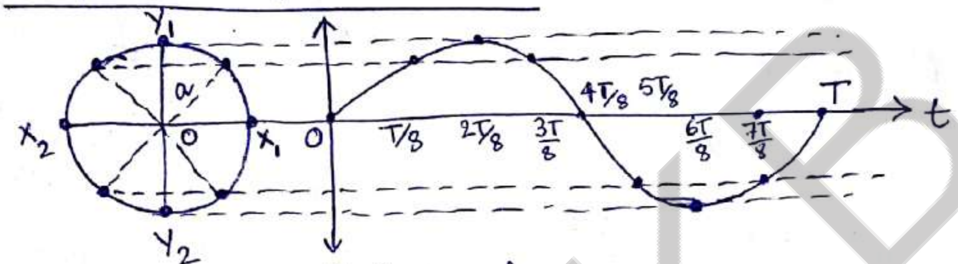
Similarly, $OB = y = OP\sin(\theta+\alpha) = a\sin(\omega t+\alpha)$

Accelaration of B is $f_B = -f_r \sin(\theta+\alpha) = -\omega^2 a \sin(\omega t+\alpha) = -\omega^2 y$

$$\therefore \underline{f_B \propto -y}.$$

$\therefore$ SHM is defined as the projection of uniform circular motion along diameter of circle.

Graphical representation

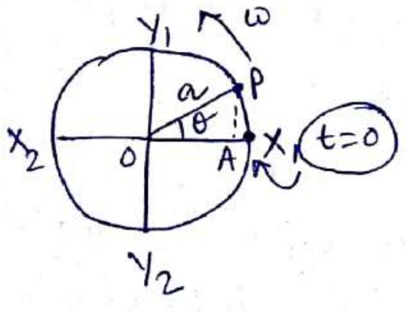


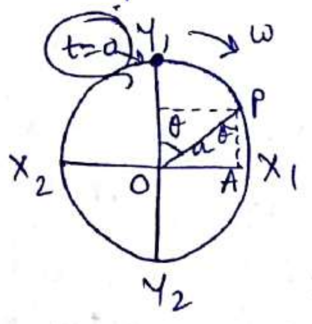
Time period = T.

$$y = a\sin\frac{2\pi}{T}t$$

(SHM along y-axis)

Displacement In SHM, displacement at time t is the distance of the particle from the mean position.





$OA = OP\cos\theta$

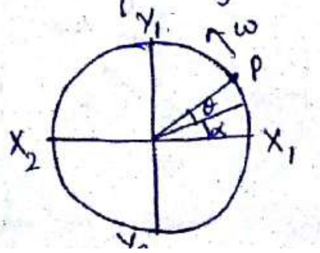
$x = a\cos\omega t$

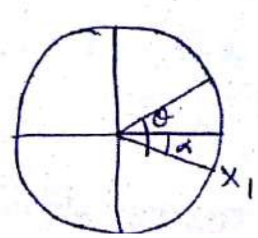
$OA = OP\cos(\pi/2-\theta)$

$x = a\sin\theta = a\sin\omega t$

Similarly, $y = a\cos\omega t$ & $y = a\sin\omega t$.

So, eq^n of SHM can be derived from any instant t.



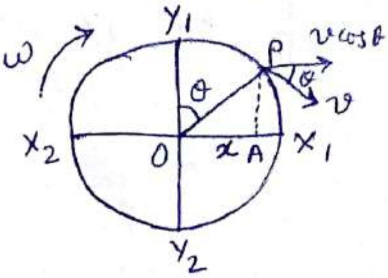


$x = a\cos(\theta + \alpha) = a\cos(\omega t + \alpha)$

Similarly, $x = a\sin(\theta+\alpha) = a\sin(\omega t + \alpha)$.

if initial position is X_1 (2^{nd} pic) then $x = a\cos(\omega t - \alpha)$

or $x = a\sin(\omega t - \alpha)$



Velocity & accelaration

velocity of SHM is component of the particle's velocity along x-axis at time t.

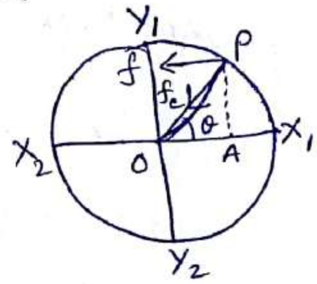
$V = a\omega$, V parallel to OA $= V\cos\theta$

$x = a\sin\theta$

$= a\omega\cos\theta = a\omega\sqrt{1-\frac{x^2}{a^2}}$

$\therefore \boxed{v = \omega\sqrt{a^2 - x^2}}$

v_{max} is at $x=0$, $v_{max} = a\omega$. & $x=a$, $v_{min}=0$.



Same with accelaration ⇒ SHM is the projection along X-axis is component of accelaration along X-axis. $f_c = -\omega^2 a$ & component aroud X_1X_2 is $-\omega^2 a\cos\theta = -\omega^2 a\cos\omega t = -\omega^2 x$.

$\therefore f = -\omega^2 x$.

$f_{max} = -\omega^2 a$ when $x = \pm a$, $f_{max} = \pm\omega^2 a$.

$f_{min} = 0$ when $x = 0$.

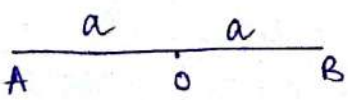
Calculus: $x = a\sin\omega t$, $v = \dot{x} = a\omega\cos\omega t = a\omega\sqrt{1-\frac{x^2}{a^2}}$

$= \omega\sqrt{a^2-x^2}$.

$f = \ddot{x} = -a\omega^2\sin\omega t = -\omega^2 x$.

$\omega^2 = f/x$ (neglect)

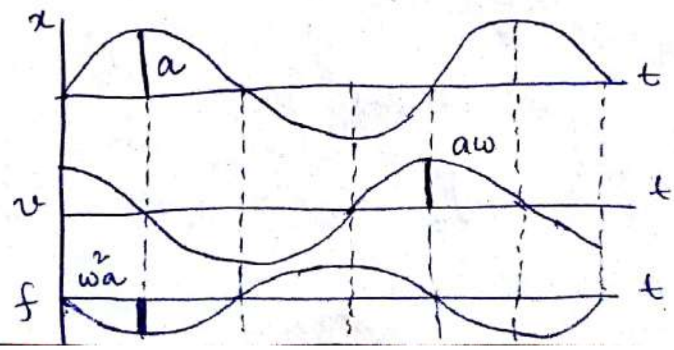
Time period $T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{a}{f}}$



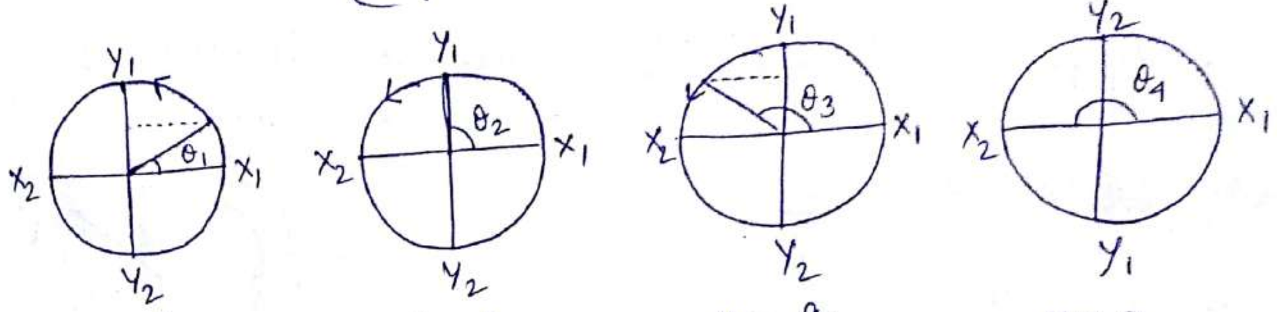
$x = a\sin\omega t = a\sin\frac{2\pi}{T}t$

$v = a\omega\cos\omega t = a\omega\cos\frac{2\pi}{T}t$

$f = -a\omega^2\sin\omega t = -a\omega^2\sin\frac{2\pi}{T}t$



Phase you see, a & ω (angular velocity) are constant.
(amplitude) $\theta = \omega t$ is changing = phase.



$y = a/2$	$y_1 = a$	$y = a/2$	$y = 0$
$\theta_1 = 30^\circ$	$\theta_2 = 90^\circ$	$\theta_3 = 150^\circ$	$\theta_4 = 180^\circ$
v upwards	$V = 0$	V= downwards	v= downwards.

phase difference 2 particles. $\phi = \theta_1 - \theta_2 = 0$ (in phase)
$= 180^\circ$ (out of phase)

Differential form & solution Homogeneous, 2nd order, ODE with constant coefficients

$F = -kx$ or $m\ddot{x} = -kx$ or $\ddot{x} + \omega^2 x = 0$, $\omega = \sqrt{\frac{k}{m}}$.

Solution : Multiply by $2\dot{x}$, $2\dot{x}\ddot{x} + 2\omega^2 x\dot{x} = 0$

Integrating $\dot{x}^2 = -\omega^2 x^2 + C$

when displacement is maximum, $x = a$, $\dot{x} = 0$ $\Rightarrow$ $C = \omega^2 a^2$

$\therefore$ $v = \dot{x} = \pm\omega\sqrt{a^2 - x^2}$

or $\pm\frac{dx}{\sqrt{a^2 - x^2}} = \omega dt$, Integrating $\sin^{-1}\frac{x}{a} = \omega t + \phi$

or $x = a\sin(\omega t + \phi)$

See, $x = a\cos(\omega t + \phi)$ also satisfy $\ddot{x} + \omega^2 x = 0$.

$x = a\sin(\omega t + \phi) = a\sin\omega t\cos\phi + a\cos\omega t\sin\phi$
$= A\sin\omega t + B\cos\omega t$.

In operator form, $\frac{d^2x}{dt^2} = D^2x$, $\frac{dx}{dt} = Dx$

$D^2x + \omega^2 x = 0$ or $D^2 = -\omega^2$ or $D = \pm i\omega$

$\therefore$ General solution $x = Ae^{i\omega t} + Be^{-i\omega t}$

For real value of x, $A = B^*$ $\quad A = a + ib$, $B = a - ib$

you can also have $x = a e^{i(\omega t + \phi)}$

Sinusoidal or cosinusoidal.

CW 1. Oscillatory motion of a particle is represented by $x = a e^{i\omega t}$. Establish the motion is SHM. Similarly if $x = a\cos\omega t + b\sin\omega t$ then SHM.

$$x = a e^{i\omega t}, \quad \dot{x} = a i \omega e^{i\omega t}, \quad \ddot{x} = -a\omega^2 e^{i\omega t} = -\omega^2 x \quad \text{(SHM)}$$

$$x = a\cos\omega t + b\sin\omega t, \quad \dot{x} = -a\omega\sin\omega t + b\omega\cos\omega t$$

$$\ddot{x} = -a\omega^2\cos\omega t - b\omega^2\sin\omega t = -\omega^2 x \quad \text{(SHM)}.$$

2. Which periodic motion is not oscillatory?

→ earth around sun or moon around earth.

3. Dimension of force constant of vibrating spring.

$$F = -Kx \qquad [K] = \frac{[\text{Force}]}{[\text{displacement}]} = \frac{[\text{Newton}]}{[\text{Metre}]}$$

also called "stiffness".

$$= \frac{[MLT^{-2}]}{[L]} = [MT^{-2}]$$

HW 1. In SHM, displacement is $x = a\sin(\omega t + \phi)$. at $t = 0$, $x = x_0$ with velocity v_0, show that $a = \sqrt{x_0^2 + \frac{v_0^2}{\omega^2}}$ & $\tan\phi = \frac{\omega x_0}{v_0}$.

2. Particle is vibrated at frequency 5 Hz in SHM. Show that when displacement exceeds 10^{-2} metre, the particle loses contact with the vibrator. Given $g = 9.8\ m/s^2$

3. In SHM, a particle has speed 80 cm/s & 60 cm/s with displacement 3 cm & 4 cm. Calculate amplitude of vibration

Energy of a particle in SHM

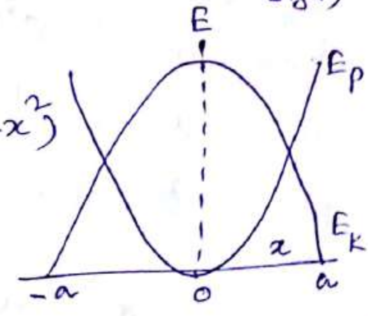
Work is done on particle to displace → restoring force. So P.E. in spring stored & motion is K.E. Total energy constant

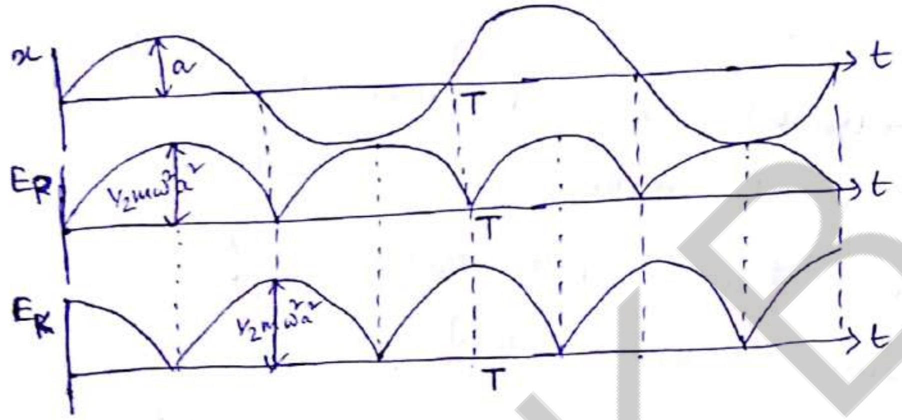
P.E. $F = mf = -m\omega^2 x \quad \therefore \; dW = F\,dx = m\omega^2 x\,dx$ (against so no -ive sign)

$\therefore \; E_p = \int_0^x m\omega^2 x\,dx = \frac{1}{2} m\omega^2 x^2.$

K.E. $v = \omega\sqrt{a^2 - x^2}, \quad E_K = \frac{1}{2} m v^2 = \frac{1}{2} m\omega^2 (a^2 - x^2)$

$E_{Tot} = E_K + E_p = \frac{1}{2} m\omega^2 a^2 = \text{constant}.$





Examples of SHM

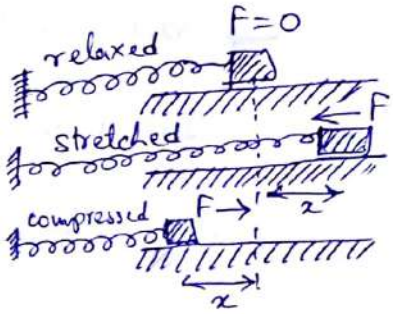
Horizontal oscillations

$F = -kx = m\ddot{x}$

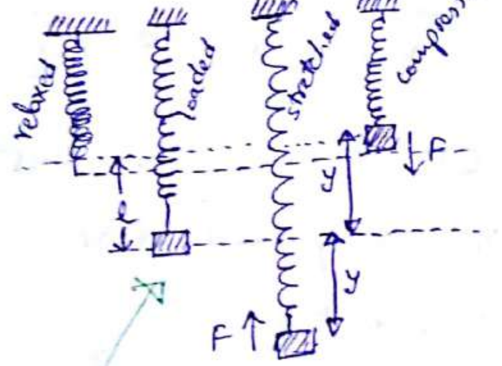
$\ddot{x} + \omega^2 x = 0 \qquad \omega = \sqrt{\frac{K}{m}}$

$x = A\cos(\omega t + \phi), \quad T = 2\pi\sqrt{\frac{m}{K}}$

initial cond^n. → material.



Vertical oscillations



static equilibrium

Tension on spring $F_0 = Kl$

Force on mass $= mg$.

Static eq. $mg = Kl$.

stretched Tension on spring $= K(l+y)$

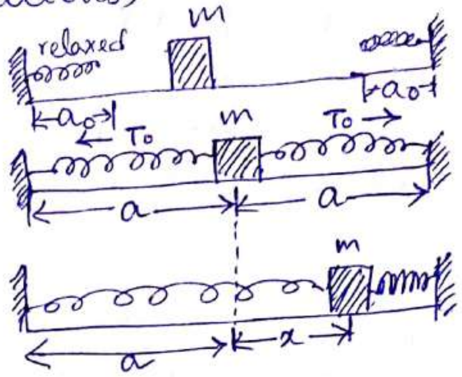
$mg - F = K(l+y) = Kl + Ky$
$= \cancel{mg} + Ky$

$F = -Ky.$

compressed $mg + F = K(l-y) = \cancel{mg} - Ky$

$F = -Ky.$

Two spring system (Longitudinal oscillations)



horizontal frictionless surface,
rigid wall, massless spring,
relaxed length a_0.

After connection, static equilibrium

$$T_0 = K(a - a_0)$$

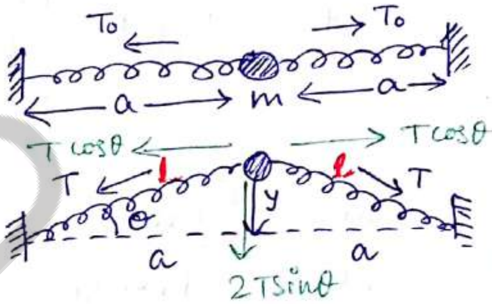
x = displacement to right. restoring force by left spring $-K(a+x-a_0)$

force on right spring $K(a-x-a_0)$

$$\therefore \; F_x = K(a-x-a_0) - K(a+x-a_0) = -2Kx$$

$$m\ddot{x} = -2Kx \quad \text{or} \quad \ddot{x} + \omega^2 x = 0 \qquad \omega = \sqrt{\frac{2K}{m}}, \quad \boxed{T_{long} = 2\pi\sqrt{\frac{m}{2K}}}$$

Two spring system (Transverse oscillations)



$$T_0 = K(a-a_0)$$

$$T = K(l - a_0)$$

$l = \sqrt{y^2 + a^2}$

$$F_y = -2T\sin\theta = -2T\frac{y}{l}$$

$$\text{or} \quad m\ddot{y} + \frac{2T}{l}y = 0 \quad \text{or} \quad \ddot{y} + \omega^2 y = 0$$

$$\omega^2 = \frac{2T}{ml} = \frac{2K(l-a_0)}{ml}, \quad \text{but } l = f(y).$$

So $\ddot{y} + \frac{2K}{m}\left(1 - \frac{a_0}{f(y)}\right)y = 0$ is not a SHM.

(a) Slinky approximation $a >> a_0$ or $\frac{a_0}{a} << 1$.

$$\omega^2 = \frac{2K}{m}\left(1 - \frac{a_0}{l}\right) = \frac{2K}{m}\left(1 - \frac{a_0}{a}\frac{a}{l}\right) \quad \text{as } l > a.$$

$$= \frac{2K}{m}. \quad \text{Then SHM.} \qquad \omega = \sqrt{\frac{2K}{m}}, \quad \boxed{T_{trans} = 2\pi\sqrt{\frac{m}{2K}}}$$

Spring

"large" harmonic oscillations

(b) small oscillation approximation $a \not\gg a_0$ but $y << a$ or l.

$$\therefore l = \sqrt{y^2 + a^2} = a\sqrt{\frac{y^2}{a^2} + 1} \sim a$$

Then also $\omega^2 = \frac{2K}{m}\left(1 - \frac{a_0}{a}\right)$ or $\boxed{T_{trans} = 2\pi\sqrt{\frac{m}{2K\left(1 - \frac{a_0}{a}\right)}}}$

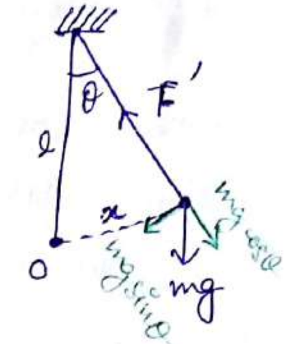
SHM.

Rubber

$$\therefore T_{long} = \sqrt{1 - \frac{a_0}{a}}\; T_{trans}.$$

So longitudinal is faster than transverse.

Simple pendulum



$F' = mg\cos\theta$
(tension in string)

$\left[\lim_{\theta\to 0}\right]$

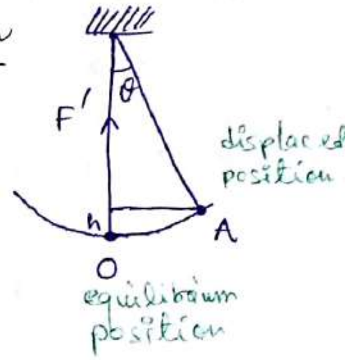
$F = -mg\sin\theta$
(restoring force) $= -mg\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots\right) \simeq -mg\theta$

$x = l\theta$

or, $m\ddot{x} = -mg\frac{x}{l}$ or $\ddot{x} + \frac{g}{l}x = 0.$

$\therefore\ \omega = \sqrt{g/l},\ T = 2\pi\sqrt{\frac{l}{g}}.$ (mass independent)

String tension when pendulum at mean position



$F' = mg + \frac{mv^2}{l}$
(centrifugal force)

at A, Energy $= KE + PE = 0 + mgh = mgh$

at O, Energy $= KE + PE = \frac{1}{2}mv^2 + 0 = \frac{1}{2}mv^2$

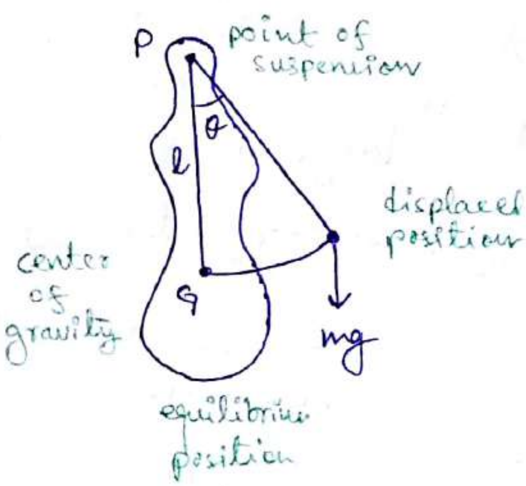
Conservation of energy $\Rightarrow \frac{1}{2}mv^2 = mgh$ or $v^2 = 2gh.$

or $v^2 = 2g(l - l\cos\theta) = 2gl(1-\cos\theta) = 2gl \times 2\sin^2\frac{\theta}{2}$

$\simeq 4gl\left(\frac{\theta}{2}\right)^2 = gl\theta^2.$

$\therefore\ F' = mg + \frac{m}{l}gl\theta^2 = mg(1+\theta^2).$

Compound Pendulum



arbitrary shaped rigid body oscillating about a horizontal axis passing through it.

restoring force $\leftrightarrow$ reactive couple or torque

moment of restoring force
$= -mgl\sin\theta$

angular acceleration $\alpha = \frac{d^2\theta}{dt^2}$, moment of inertia $= I$.

$$\tau = I\alpha = I\frac{d^2\theta}{dt^2} = -mgl\sin\theta$$

or $$\frac{d^2\theta}{dt^2} = -\frac{mgl}{I}\sin\theta \simeq -\frac{mgl}{I}\theta \quad \text{or} \quad \frac{d^2\theta}{dt^2} + \omega^2\theta = 0$$

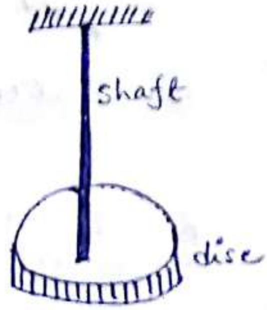
$$T = 2\pi\sqrt{\frac{I}{mgl}}$$

If we consider moment of inertia about a parallel axis through G,
K = radius of gyration then using parallel axis theorem,

$$I = mK^2 + ml^2 \quad \therefore T = 2\pi\sqrt{\frac{K^2/l + l}{g}} = 2\pi\sqrt{\frac{l'}{g}}.$$

equivalent length of simple pendulum $= \frac{K^2}{l} + l$.

Torsional Pendulum



twist of shaft → torsional oscillations

torsional couple $= -\tau\theta$

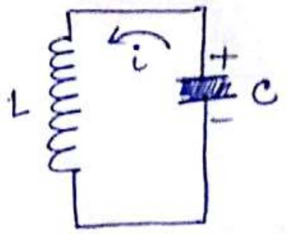
couple due to accelaration $= I\frac{d^2\theta}{dt^2}$

$$I\frac{d^2\theta}{dt^2} = -\tau\theta \quad , \quad T = 2\pi\sqrt{\frac{I}{\tau}}$$

From classical mechanics course, $\tau = \frac{\pi\eta d^4}{32l} = \frac{\pi\eta r^4}{2l}$

$d =$ shaft diameter, $\eta =$ modulus of rigidity,
$= 2r$.

Electrical Oscillator



Capacitor is charged ⇒ electrostatic energy in dielectric media. It discharges through the inductor electrostatic energy ⇔ magnetic energy. (no dissipation of heat)

voltage across inductor $= -L\frac{di}{dt} = -L\frac{d^2q}{dt^2}$ $\quad q =$ charge

voltage across capacitor $= \frac{q}{C}$.

No e.m.f. circuit, $\frac{q}{C} = -L\frac{d^2q}{dt^2}$ or $\frac{d^2q}{dt^2} + \frac{q}{LC} = 0$

$\omega^2 = \frac{1}{LC}$, $q = q_0\sin(\omega t + \phi)$. charge on capacitor varies harmonically.

$$i = \frac{dq}{dt} = \omega q_0 \cos(\omega t + \phi)$$

$$V = \frac{q}{C} = \frac{q_0}{C} \sin(\omega t + \phi)$$

Total energy = magnetic energy + electric energy

$$= \int iV\,dt + \frac{1}{2}CV^2 = \int i L \frac{di}{dt}\,dt + \frac{1}{2}CV^2$$

$$= \int L i\,di + \frac{1}{2}CV^2 = \frac{1}{2}Li^2 + \frac{1}{2}CV^2 = \frac{1}{2}L\dot{q}^2 + \frac{1}{2}CV^2$$

In mechanical oscillation, Total energy $= \frac{1}{2} m \dot{x}^2 + \frac{1}{2} k x^2$

equivalence

$$\frac{1}{2}CV^2 = \frac{1}{2}C\left(\frac{q}{C}\right)^2 = \frac{q^2}{2C}$$

In electrical oscillation, Total energy $= \frac{1}{2} L \dot{q}^2 + \frac{1}{2C} q^2$

Resultant / Superposition of Harmonic oscillations

The resultant of two or more harmonic displacements is the algebraic sum of individual displacements. For linear homogeneous differential equations, sum of any two solutions is also a solution.

Realize that if $\frac{d^2x}{dt^2} = -\omega^2 x + \alpha x^2 + \beta x^3 + \cdots$ then if

$$\frac{d^2x_1}{dt^2} = -\omega^2 x_1 + \alpha x_1^2 + \beta x_1^3 + \cdots \ \& \ \frac{d^2x_2}{dt^2} = -\omega^2 x_2 + \alpha x_2^2 + \beta x_2^3 + \cdots$$

then $x_1 + x_2$ isn't a solution because if $x_1 + x_2 = x_3$. then

$$\frac{d^2x_1}{dt^2} + \frac{d^2x_2}{dt^2} = -\omega^2(x_1+x_2) + \alpha(x_1^2 + x_2^2) + \beta(x_1^3 + x_2^3) + \cdots$$

$$\therefore \frac{d^2x_3}{dt^2} = -\omega^2 x_3 + \alpha(x_3^2 - 2x_1x_2) + \beta(x_3^3 - 3x_1^2x_2 - 3x_1x_2^2) + \cdots$$

Composition of two colinear SHM of same frequency but different amplitude & phase:

Frequency $\omega = 2\pi\nu$, amplitude a & b, phase difference ϕ

$$x_1 = a\sin\omega t, \quad x_2 = b\sin(\omega t + \phi)$$

Time period for both motion is same & so phase difference is also same.

resultant displacement $x = x_1 + x_2 = a\sin\omega t + b\sin(\omega t + \phi)$

$= (a + b\cos\phi)\sin\omega t + b\sin\phi\cos\omega t = A\cos\theta\sin\omega t + A\sin\theta\cos\omega t$

$x = A\sin(\omega t + \theta) \Rightarrow$ S. H. M.

Amplitude of resultant wave $A^2 = (a + b\cos\phi)^2 + b^2\sin^2\phi$

or $A = (a^2 + b^2 + 2ab\cos\phi)^{1/2}$

phase of resultant wave $\tan\theta = \dfrac{b\sin\phi}{a + b\cos\phi}$

$\therefore \quad x = \sqrt{a^2+b^2+2ab\cos\phi}\,\sin\left(\omega t + \tan^{-1}\left\{\dfrac{b\sin\phi}{a+b\cos\phi}\right\}\right)$

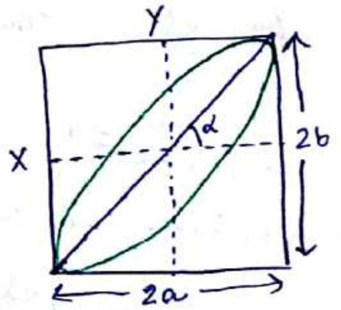
if $\phi = 0$ then $\theta = 0$, $A = a+b$, $x = (a+b)\sin\omega t$

if $\phi = \pi$ then $\theta = 0$ (opposite phase), $A = a-b$, $x = (a-b)\sin\omega t$.

if $a=b$, $x = 0 \Rightarrow$ no resultant motion.

Composition of two SHM at right angle with same frequency but different in phase & amplitude

Again, say two SHM acting in X & Y axis, amplitude a & b, phase difference ϕ.



$x = a\sin\omega t$, $y = b\sin(\omega t + \phi)$

$\therefore \cos\omega t = \sqrt{1 - x^2/a^2}$

and $\sin\omega t\cos\phi + \cos\omega t\sin\phi = y/b$.

or $\dfrac{x}{a}\cos\phi + \sqrt{1 - \dfrac{x^2}{a^2}}\sin\phi = \dfrac{y}{b}$

or $\left(\dfrac{y}{b} - \dfrac{x}{a}\cos\phi\right)^2 = \left(1 - \dfrac{x^2}{a^2}\right)\sin^2\phi$

or $\boxed{\dfrac{y^2}{b^2} + \dfrac{x^2}{a^2} - \dfrac{2xy}{ab}\cos\phi = \sin^2\phi}$ ——— (1)

This is equation of ellipse confined to rectangle of side 2a & 2b with direction of major axis $\tan\alpha = \dfrac{2ab}{a^2 - b^2}\cos\phi$.

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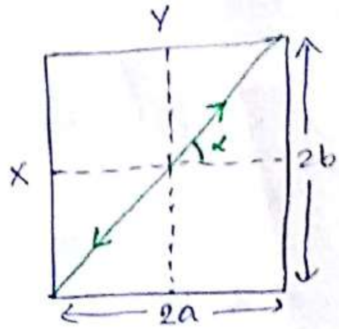
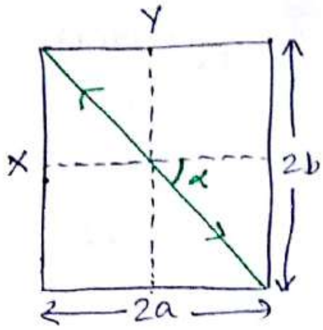
(a) $\phi = 0$ $\quad \sin\phi = 0, \cos\phi = 1, \quad \frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{2xy}{ab} = 0$

or $\left(\frac{y}{b} - \frac{x}{a}\right)^2 = 0$ $\quad$ or $y = \frac{b}{a}x$



Straight line passing through origin & inclined to x-axis at angle $\alpha = \tan^{-1}\frac{b}{a}$ & with resultant amplitude $= \sqrt{a^2+b^2}$

(b) $\phi = \pi$ $\quad$ Two motions are in opposite phase



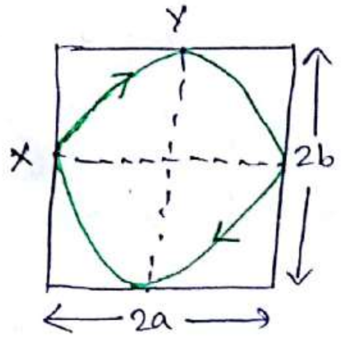
Then the combined equation is

$$\frac{y^2}{b^2} + \frac{x^2}{a^2} + \frac{2xy}{ab} = 0 \quad \text{or} \quad \left(\frac{y}{b} + \frac{x}{a}\right)^2 = 0$$

$\therefore y = -\frac{b}{a}x.$

straight line passing through origin & inclined to x-axis at angle $\tan\alpha = -\frac{b}{a}$. If $a = b$, $\alpha = 135°$

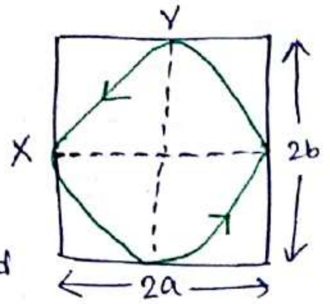
(c) $\phi = \pi/2$ $\quad$ Then the combined equation is



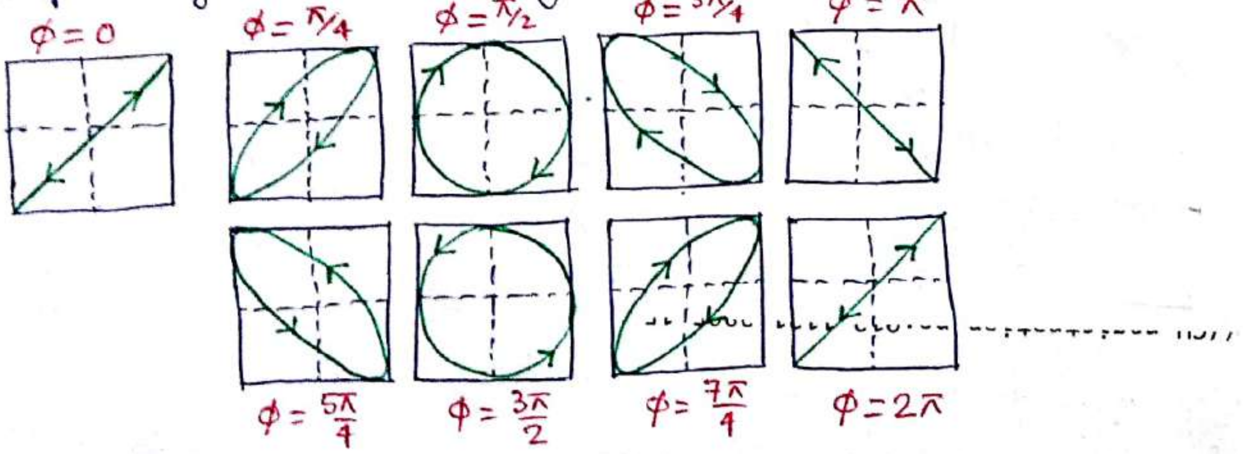
$\frac{y^2}{b^2} + \frac{x^2}{a^2} = 1$, elliptical motion with major axis $2a$, minor axis $2b$.

If $a = b$, then circular motion with $x^2 + y^2 = a^2$

(d) $\phi = \frac{3\pi}{2}$ $\quad$ Then the combined equation is



$\frac{y^2}{b^2} + \frac{x^2}{a^2} = 1$, elliptic motion but counter-clockwise. In ray optics, this is called left-handed elliptically polarized light/vibration.

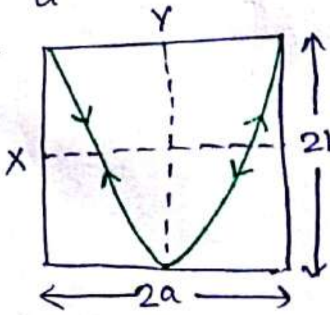


Composition of two SHM at right angle with different frequency, different phase, different amplitude:

Complicated motion $\longrightarrow$ Lissajous figures. Suppose frequencies are in 1 : 2 ratio $x = a\cos\omega t$, $y = b\cos(2\omega t + \phi)$

$$\therefore \frac{y}{b} = \cos(2\omega t)\cos\phi - \sin(2\omega t)\sin\phi$$
$$= (2\cos^2\omega t - 1)\cos\phi - 2\sin\omega t\cos\omega t\sin\phi.$$
$$= \left(2\frac{x^2}{a^2} - 1\right)\cos\phi - 2\frac{x}{a}\sqrt{1-\frac{x^2}{a^2}}\sin\phi.$$
$$\text{or } \left(\frac{y}{b} + \cos\phi\right) - \frac{2x^2}{a^2}\cos\phi = -\frac{2x}{a}\sqrt{1-\frac{x^2}{a^2}}\sin\phi.$$
$$\text{or } \left(\frac{y}{b} + \cos\phi\right)^2 + \frac{4x^2}{a^2}\left(\frac{x^2}{a^2} - 1 - \frac{y}{b}\cos\phi\right) = 0 \Rightarrow 4^{th} \text{ degree equation}$$

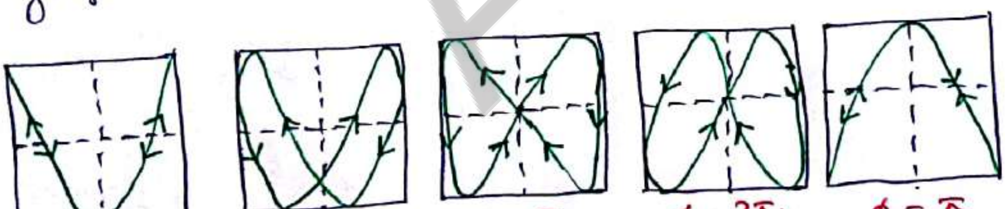
$\phi = 0$ $\quad \left(\frac{y}{b}+1\right)^2 + \frac{4x^2}{a^2}\left(\frac{x^2}{a^2} - 1 - \frac{y}{b}\right) = 0$ or $\left(\frac{y}{b} - \frac{2x^2}{a^2} + 1\right)^2 = 0$



Two coincident parabola with vertex at $(0, -b)$ with equation $\frac{y}{b} - \frac{2x^2}{a^2} + 1 = 0$

$$\text{or } x^2 = \frac{a^2}{2b}(y+b).$$

$\phi \neq 0$ very complex to resolve analytically & graphical method is the most convenient method.



$\phi = 0$ $\quad \phi = \pi/4$ $\quad \phi = \pi/2$ $\quad \phi = 3\pi/4$ $\quad \phi = \pi$

frequency ratio 1:2

$\phi = 0$ $\quad \phi = \pi/4$ $\quad \phi = \pi/2$ $\quad \phi = \frac{3\pi}{4}$ $\quad \phi = \pi$

frequency ratio 1:3

So frequencies need to be in a commensurate ratio to give a periodic motion. Notice the interesting features that (1) resultant curve is always inside rectangle & the motion is periodic, (2) Number of tangential point in $x : y$ is the frequency ratio inverse.

HW 1. A particle is simultaneously subjected to two SHM in same direction, each of frequency 5 Hz. If amplitudes are 0.005 m & 0.002 m & phase difference is 45°, find the amplitude of the resultant ~~direction~~ displacement & its phase relative to the first component. Write down the expression for the resultant displacement as a function of time.

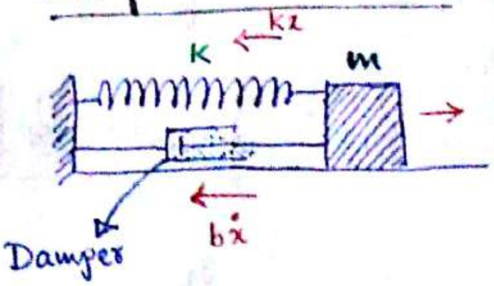
2. Two vibrations along the same line are described by

$x_1 = 0.03 \cos 10\pi t$, $x_2 = 0.03 \cos 12\pi t$, x_1, x_2 in metres & t in seconds. Obtain the equation describing the resultant motion and the beat period (beat period is the time interval between two consecutive maximum amplitude).

Free Damped harmonic motion

Damping of a real system is a complex phenomena involving several kind of damping force. Damping force of a body in a fluid is a function of velocity. This is called "viscous damping." When an oscillating body is contact with a surface, the frictional force is called "Coulomb friction". Also in solids, energy is partly lost due to internal friction & imperfect elasticity of the material. Experiments suggest that such resistive force is independent of frequency & proportional to amplitude. This is called "structural damping." The viscouse damping force may be represented as $F = -Av + \cancel{Bv^2} - \cancel{Cv^3} \to 0 + \cdots$ and such approximation is "linear damping".

Damped oscillation of a system with 1 degree of freedom



inertial force $m\ddot{x}$ is balanced by elastic restoring force kx & viscous damping force $b\dot{x}$

$$\therefore\ m\ddot{x} = -b\dot{x} - kx \quad \text{or} \quad \frac{d^2x}{dt^2} + \underbrace{\frac{b}{m}}_{=\gamma}\frac{dx}{dt} + \underbrace{\frac{K}{m}}_{=\omega_0^2}x = 0.$$

This is a linear homogeneous 2^{nd} order ODE.

Let the trial solution $x = Ae^{\alpha t}$, substituting we get

$$(\alpha^2 + \gamma\alpha + \omega_0^2)Ae^{\alpha t} = 0 \quad \Rightarrow \quad \alpha^2 + \gamma\alpha + \omega_0^2 = 0.$$

$$\therefore\ \alpha = \frac{-\gamma \pm \sqrt{\gamma^2 - 4\omega_0^2}}{2} = -\frac{\gamma}{2} \pm \sqrt{\gamma^2/4 - \omega_0^2}$$

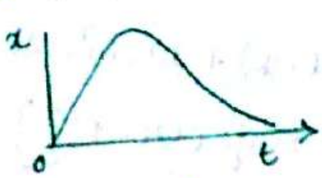
$$\therefore\ \text{Solution}\quad x = A_1 \exp\left[-\frac{\gamma}{2} + \sqrt{\frac{\gamma^2}{4} - \omega_0^2}\right]t + A_2 \exp\left[-\frac{\gamma}{2} - \sqrt{\frac{\gamma^2}{4} - \omega_0^2}\right]t$$

$$= e^{-\gamma t/2}\left[A_1 \exp\left(\sqrt{\gamma^2/4 - \omega_0^2}\,t\right) + A_2 \exp\left(-\sqrt{\frac{\gamma^2}{4} - \omega_0^2}\,t\right)\right]$$

We can have three possibilities :

(a) Heavy damping $\gamma^2/4 > \omega_0^2$ $\quad q = \sqrt{\gamma^2/4 - \omega_0^2} > 0$.

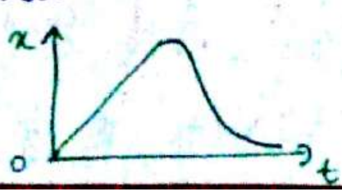
$x = e^{-\gamma t/2}(A_1 e^{qt} + A_2 e^{-qt})$. This means that x cannot be negative and at $t \approx 0$, $e^{-\gamma t/2} \approx 1$ & e^{qt} contributes like exponenti then at $t \to \infty$, it'll damp to x (initial). If we had started at $x = 0$, after a time interval it decays back to zero ⇒ "Dead beat Galvanometer." no oscillation



(b) Critical damping $\gamma^2/4 = \omega_0^2$: $x = (A_1 + A_2)e^{-\gamma t/2}$. The damping is slower but it has a discrepancy that at $x=0$ at $t=0$, $v=0$ which is not true. Changing the trial solution, we can derive

$x \sim t\,e^{-\gamma t/2}$ mean at $t \approx 0$, $e^{-\gamma t/2} \approx 1$ & $x \propto t$ & later $t \to \infty$, $e^{-\gamma t/2}$ dominates. x is never negative ⇒ no oscillati

"pointer_type galvanometer"



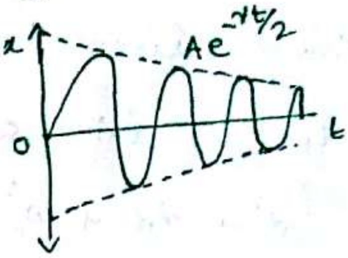
(c) Weak damping $\frac{\gamma^2}{4} < \omega_0^2$ $\quad q = \sqrt{\frac{\gamma^2}{4} - \omega_0^2}$ = imaginary.

This gives oscillatory damped harmonic motion

$$x = e^{-\gamma t/2}\left[A_1 e^{i\sqrt{\omega_0^2 - \gamma^2/4}\, t} + A_2 e^{-i\sqrt{\omega_0^2 - \gamma^2/4}\, t}\right] \qquad \boxed{\omega = \sqrt{\omega_0^2 - \gamma^2/4}}$$

$$= e^{-\gamma t/2}\left(A_1 e^{i\omega t} + A_2 e^{-i\omega t}\right)$$

$$= e^{-\gamma t/2}\Big[\underbrace{(A_1 + A_2)}_{A\cos\delta}\cos\omega t + \underbrace{i(A_1 - A_2)}_{A\sin\delta}\sin\omega t\Big] = A e^{-\gamma t/2}\cos(\omega t - \delta)$$



Amplitude decreases in due time

Angular frequency is less than undamped motion.

$\tau = 2/\gamma$ = mean life time of oscillation.

Energy of a weakly damped oscillator

Using $x = A e^{-\gamma t/2}\cos(\omega t - \delta)$ we develop expression for average energy.

$$\dot{x} = -\frac{\gamma}{2} A e^{-\gamma t/2}\cos(\omega t - \delta) - A e^{-\gamma t/2}\,\omega\sin(\omega t - \delta)$$

$\therefore$ Kinetic energy (instantaneous) of the vibrating body

$$\frac{1}{2} m\dot{x}^2 = \frac{1}{2} m A^2\left[\frac{\gamma^2}{4}\cos^2(\omega t - \delta) + \omega^2\sin^2(\omega t - \delta) + \gamma\omega\cos(\omega t - \delta)\sin(\omega t - \delta)\right]$$

Potential energy $= \int_0^x F\,dx = \int_0^x Kx\,dx = \frac{1}{2}Kx^2 = \frac{1}{2}KA^2 e^{-\gamma t}\cos^2(\omega t - \delta)$

$$= \frac{1}{2} m\omega_0^2 A^2 e^{-\gamma t}\cos^2(\omega t + \delta)$$

$\therefore$ Total energy = KE + PE =

$$\frac{1}{2} m A^2 e^{-\gamma t}\left[\frac{\gamma^2}{4}\cos^2(\omega t - \delta) + \omega^2\sin^2(\omega t - \delta) + \omega_0^2\cos^2(\omega t - \delta) + \frac{\gamma\omega}{2}\sin\{2(\omega t - \delta)\}\right]$$

For small damping, $\gamma << 2\omega_0$, then $e^{-\gamma t}$ does not change appreciably during one time period $T = \frac{2\pi}{\omega}$, then time averaged energy of the oscillator is

$$\langle E\rangle = \frac{1}{2} m A^2 e^{-\gamma t}\left[\frac{\gamma^2}{4}\langle\cos^2(\omega t - \delta)\rangle + \omega^2\langle\sin^2(\omega t - \delta)\rangle + \omega_0^2\langle\cos^2(\omega t - \delta)\rangle + \frac{\gamma\omega}{2}\langle\sin\{2(\omega t - \delta)\}\rangle\right]$$

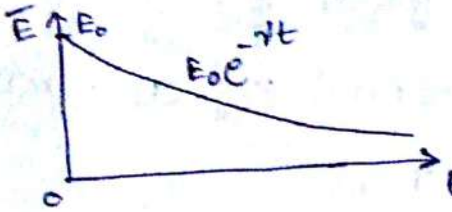
$$\langle\sin\{2(\omega t - \delta)\}\rangle \to 0$$

Now $\langle\cos^2(\omega t - \delta)\rangle = \frac{1}{2\pi}\int_0^{2\pi}\cos^2(\omega t - \delta)\,d(\omega t) = \frac{1}{2\pi}\frac{1}{2}\int_0^{2\pi} 2\cos^2 x\,dx$

$$= \frac{1}{4\pi}\int_0^{2\pi}(1 + \cos 2x)\,dx = \frac{1}{2} = \langle\sin^2(\omega t - \delta)\rangle$$

$\therefore \langle E \rangle = \frac{1}{2} m A^2 e^{-\gamma t} \left[\cancel{\frac{\gamma^2}{8}} + \left(\omega_0^2 - \cancel{\frac{\gamma^2}{4}}\right)\frac{1}{2} + \frac{\omega_0^2}{2} \right] = \frac{1}{2} m \omega_0^2 A^2 e^{-\gamma t}$

$\boxed{\langle E \rangle = E_0 e^{-\gamma t}}$ where $E_0 = \frac{1}{2} m \omega_0^2 A^2$ is energy of undamped oscillator

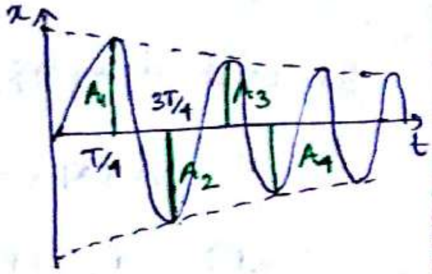


The average power dissipation in one time period

$\langle P(t) \rangle = \frac{d}{dt} \langle E(t) \rangle = \gamma \langle E(t) \rangle$. due to friction

Estimation of Damping

There are various ways of estimation of the damping of an oscillator. Let us choose initial condition at $t=0$, $x=0$, $\frac{dx}{dt} = v_0$ and $\delta = \pi/2$, $x = A e^{-\gamma t/2} \cos(\omega t - \pi/2) = A e^{-\gamma t/2} \sin \omega t$



Logarithmic Decrement

$x = A e^{-\gamma t/2} \sin \omega t = A e^{-\gamma t/2} \sin \frac{2\pi t}{T}$

at $t = \frac{T}{4}$, $x_1^{max} = A e^{-\gamma T/8} \sin \frac{2\pi}{T}\frac{T}{4} = A e^{-\gamma t/8}$

at $t = \frac{3T}{4}$, $x_2^{max} = A e^{-3\gamma T/8}$

at $t = \frac{5T}{4}$, $x_3^{max} = A e^{-5\gamma T/8}$ etc.

$\therefore \frac{x_1^{max}}{x_2^{max}} = \frac{x_2^{max}}{x_3^{max}} = \frac{x_3^{max}}{x_4^{max}} = \cdots = \frac{x_{n-1}^{max}}{x_n^{max}} = e^{\gamma T/4} = d$ (constant)

"d" is called decrement of the motion. $\lambda = \ln d$ is the logarithmic decrement of the motion $= \ln e^{\gamma T/4} = \frac{\gamma T}{4}$

$\therefore \frac{x_1^{max}}{x_2^{max}} = \frac{x_2^{max}}{x_3^{max}} = \cdots = \frac{x_{n-1}^{max}}{x_n^{max}} = e^{\lambda}$

Multiplying, $\frac{x_1^{max}}{x_n^{max}} = e^{(n-1)\lambda}$ or $\lambda = \frac{1}{n-1} \ln\left(\frac{x_1^{max}}{x_n^{max}}\right)$

$\lambda = \frac{2.303}{n-1} \log_{10}\left(\frac{x_1^{max}}{x_n^{max}}\right)$

This method is used to determine the corrected last throw of a Ballistic galvanometer due to damping.

Relation between undamped throw θ_0 & first throw θ_1 is

$\theta_1 = \theta_0 e^{-\gamma T/8}$ $\therefore \theta_0 = \theta_1 e^{\gamma T/8} = \theta_1 e^{\lambda/2} \simeq \theta_1 (1 + \frac{\lambda}{2})$ for $\lambda \ll 1$

So knowing λ, we can correct θ_1 for damping.

Quality Factor (Q-Value)

Another method to express damping in an oscillatory system is to measure the rate of decay of energy. Quality factor $Q = \frac{\omega}{\gamma}$ $= \frac{\omega_0}{\gamma}\sqrt{1-\gamma^2/4\omega_0^2}$. while $\langle E\rangle = E_0 e^{-\gamma t}$, power $\langle P(t)\rangle = \frac{d}{dt}\langle E\rangle = \gamma\langle E\rangle$

So the average energy dissipated in time period T is

$$\gamma T\langle E\rangle = \frac{2\pi\gamma}{\omega}\langle E\rangle = \frac{2\pi}{Q}\langle E\rangle = \frac{2\pi}{Q} \times \text{average energy stored.}$$

$$\therefore \boxed{Q = 2\pi \times \frac{\text{Average energy stored in one time period}}{\text{Average energy lost in one time period}}}$$

In weak damping limit $\frac{\gamma^2}{4\omega_0^2} << 1$, $Q = \frac{\omega_0}{\gamma}$. As $\gamma \to 0$, $Q \to \infty$

$\therefore$ $x = A\exp(-\frac{\omega_0 t}{2Q})\cos(\omega_0 t - \delta)$ in limit $\gamma^2/4\omega_0^2 << 1$

$\langle E\rangle = E_0 \exp(-\frac{\omega_0 t}{Q})$ and see that $\tau_1 = \frac{Q}{\omega_0}$, $\langle E\rangle = E_0 e^{-1}$

and no. of complete oscillation if is n, then $n = \frac{\omega_0}{2\pi}\tau_1 = \frac{Q}{2\pi}$

So $\langle E\rangle$ reduces to e^{-1} of $\langle E\rangle$ in $Q/2\pi$ cycles of oscillation.

Note that $\boxed{\lambda = \frac{\gamma T}{4},\ \tau = \frac{2}{\gamma}\ \&\ Q = \frac{\omega_0}{\gamma},\ \tau_1 = \frac{Q}{\omega_0} = \frac{1}{\gamma}}$

"Moving coil Galvanometer" is the example of damped harmonic motion. Similarly, current or charge oscillation in LCR circuit, mechanical vibration of a string or tuning fork etc.

Forced Vibration

Vibrating system with damping + periodic force = forced vibration natural vibration dies out, system tunes to the frequency of force. For example, a bridge vibrates in the influence of marching soldiers. Contribution are restoring force kx, damping force $b\dot{x}$, inertial force $m\ddot{x}$ & external periodic force $F(t) = F_0\cos\omega t$.

$\therefore$ Equation of motion of the body is

$$m\frac{d^2x}{dt^2} = -b\frac{dx}{dt} - kx + F(t)$$

or $\frac{d^2x}{dt^2} + \gamma\frac{dx}{dt} + \omega_0^2 x = f_0\cos\omega t$, $\gamma = \frac{b}{m}$, $\omega_0^2 = \frac{k}{m}$, $f_0 = \frac{F_0}{m}$.

linear homogeneous 2nd order ODE. Solution of this we can separate out as $\frac{d^2x_1}{dt^2} + \gamma\frac{dx_1}{dt} + \omega_0^2 x_1 = f_0\cos\omega t$ & $\frac{d^2x_2}{dt^2} + \gamma\frac{dx_2}{dt} + \omega_0^2 x_2 = 0$ so that $x_1 + x_2$ is a solution. Now we know $x_2 = Ae^{-\gamma t/2}\cos(\omega^* t - \delta)$ with $\omega^* = \sqrt{\omega_0^2 - \gamma^2/4}$ & will die out in time. (transient state). For x_1, we can write $x_1 = B\cos(\omega t - \delta)$ where B & δ are to be determined

$X = Re(Be^{i(\omega t - \delta)})$. In this notation,

$$\frac{d^2X}{dt^2} + \gamma\frac{dX}{dt} + \omega_0^2 X = f_0 e^{i\omega t} = f_0 e^{i(\omega t - \delta)} e^{i\delta}$$

or $\left[B\left[(\omega_0^2 - \omega^2) + i\omega\gamma\right] - f_0 e^{i\delta}\right] e^{i(\omega t - \delta)} = 0$, $\forall t$.

$B(\omega_0^2 - \omega^2 + i\omega\gamma) - f_0 e^{i\delta} = 0$ or $Be^{-i\delta} = \frac{f_0}{\omega_0^2 - \omega^2 + i\omega\gamma}$

or $B\cos\delta - iB\sin\delta = \frac{f_0[\omega_0^2 - \omega^2 - i\omega\gamma]}{(\omega_0^2 - \omega^2)^2 + \omega^2\gamma^2}$

$\therefore B\cos\delta = \frac{f_0(\omega_0^2 - \omega^2)}{(\omega_0^2 - \omega^2)^2 + \omega^2\gamma^2}$, $B\sin\delta = \frac{f_0\omega\gamma}{(\omega_0^2 - \omega^2)^2 + \omega^2\gamma^2}$ $\therefore B = \frac{f_0}{\sqrt{(\omega_0^2 - \omega^2)^2 + \omega^2\gamma^2}}$

$\delta = \tan^{-1}\left(\frac{\omega\gamma}{\omega_0^2 - \omega^2}\right)$

$$\therefore \boxed{x_1 = \frac{f_0}{\sqrt{(\omega_0^2 - \omega^2)^2 + \omega^2\gamma^2}}\cos\left(\omega t - \tan^{-1}\left(\frac{\omega\gamma}{\omega_0^2 - \omega^2}\right)\right)}$$

Steady state solution

Its dependent on $F_0, m, \omega, \omega_0, \gamma$ & there is a phase difference δ between force & displacement. When $D = (\omega_0^2 - \omega^2)^2 + \omega^2\gamma^2$ is minimum B is maximum amplitude. If this frequency is ω_r then $\left.\frac{dD}{d\omega}\right|_{\omega = \omega_r} = 0$ and $\left.\frac{d^2D}{d\omega^2}\right|_{\omega = \omega_r} > 0$. $\therefore -2(\omega_0^2 - \omega_r^2)2\omega_r + 2\omega_r\gamma^2 = 0$

or $\omega_r = \sqrt{\omega_0^2 - \gamma^2/2}$ and convince yourself $\frac{d^2D}{d\omega^2} > 0$ if $\frac{\gamma^2}{2} < \omega_0^2$.

Thus amplitude of forced oscillation is maximum if frequency of the driving force is nearly equal to frequency of natural oscillation

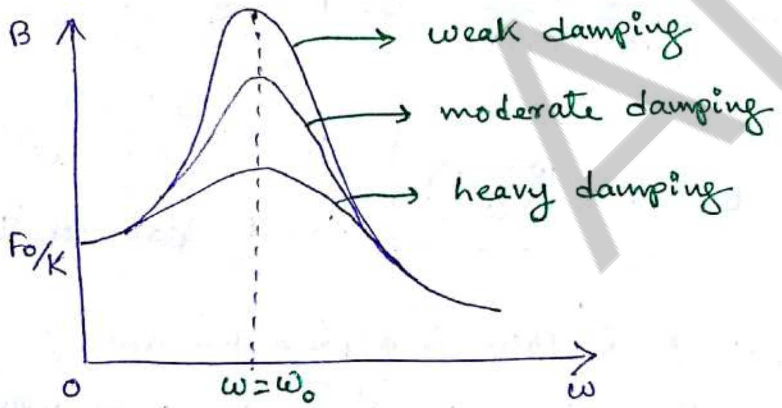
At $\omega = \omega_r$, $B_{max} = \dfrac{f_0}{\gamma(\omega_0^2 - \gamma^2/4)^{1/2}}$ and $\gamma << \omega_0$, $B_{max} \approx \dfrac{f_0}{\gamma\omega_0} = \dfrac{F_0}{m\gamma\omega_0} = \dfrac{F_0}{b\omega_0}$

Thus in this limit $\omega_r \simeq \omega_0$ and the amplitude is controlled by "b" and the forced oscillator is "**resistance controlled.**"

Recall $B = \dfrac{F_0/m}{\sqrt{(\omega_0^2-\omega^2)^2 + \omega^2\gamma^2}}$, In limit $\omega << \omega_0$, $B \approx \dfrac{F_0/m}{\omega_0^2\sqrt{1+\frac{\omega^2}{\omega_0^2}\frac{\gamma^2}{\omega_0^2}}} \simeq \dfrac{F_0}{m\omega_0^2} = \dfrac{F_0}{K}$

This displacement a constant force F_0 would produce. when $\omega \to 0$, $F(t) \to F_0$ or we get back $m\dfrac{d^2x}{dt^2} = -m\omega^2 x$ very small role than Kx term. ∴ Response of the oscillator is controlled by the stiffness constant K & the oscillator is "**stiffness controlled.**"

Similarly for $\omega >> \omega_0$, $B \simeq \dfrac{F_0/m}{\omega^2\sqrt{1+\frac{\gamma^2}{\omega_0^2}\frac{\omega_0^2}{\omega^2}}}$ which for weak damping $\gamma << \omega_0$ is $B \simeq \dfrac{F_0}{m\omega^2}$ and $m\omega^2 x$ is dominating and the oscillator is "**mass or inertia controlled**."



amplitude resonance at $\omega = \omega_0$ when $\gamma^2/2 < \omega_0^2$.

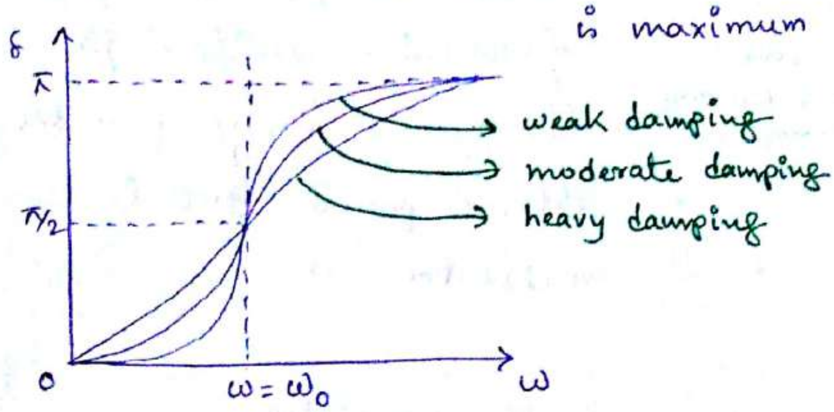
Also when $\omega << \omega_0$,

$$\tan\delta = \frac{\omega\gamma}{\omega_0^2 - \omega^2} \simeq \frac{\omega}{\omega_0}\frac{\gamma}{\omega_0}$$

as $\omega \to 0$, $\delta \to 0$. Thus for low frequency of driving force, displacement is nearly inphase with driving force. If $\omega >> \omega_0$, $\tan\delta \simeq -\dfrac{\gamma}{\omega} \simeq \dfrac{\gamma}{\omega_0}\dfrac{\omega_0}{\omega}$ which for weak damping $\gamma << \omega_0$ has small negative value or $\underline{\delta \simeq \pi}$.

∴ If frequency of driving force >> natural frequency of free oscillations, then displacement will be out of phase with driving force. Also ~~when~~ ~~[illegible]~~ accelaration will be inphase with driving force.

But at resonance, $\omega \simeq \omega_0$ & $\tan\delta = \infty$ $\therefore$ $\delta = \pi/2$ or displacement is maximum when driving force is zero.



Displacement x_1 lags the force $F(t)$ by δ.

<u>Velocity Resonance</u> $\quad x_1 = B\cos(\omega t - \delta)$ $\therefore$ $\dot{x}_1 = -\omega B \sin(\omega t - \delta)$

or $v = v_0 \cos(\omega t - \phi)$ where $v_0 = \omega B = \dfrac{F_0/m}{\sqrt{\dfrac{(\omega_0^2 - \omega^2)^2}{\omega^2} + \gamma^2}}$

$= v_0 \cos(\omega t - \delta + \pi/2)$

and $\phi = \delta - \pi/2$. $[-\sin(\omega t - \delta) = \cos(\omega t - \delta + \pi/2)]$

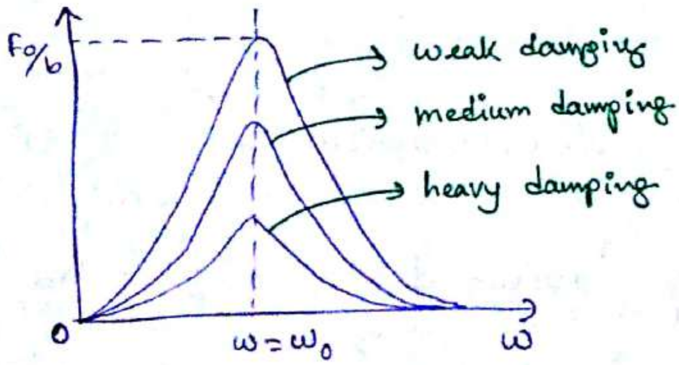
$\therefore$ Velocity leads the displacement in phase by $\pi/2$. V_0 is maximum when denominator is minimum. $\dfrac{d}{d\omega}\left[\dfrac{(\omega_0^2 - \omega^2)^2}{\omega^2} + \gamma^2\right]\Bigg|_{\omega = \omega_r} = 0$

or $\omega_r = \omega_0$. So at $\omega = \omega_0$, V_0 is maximum, velocity resonance

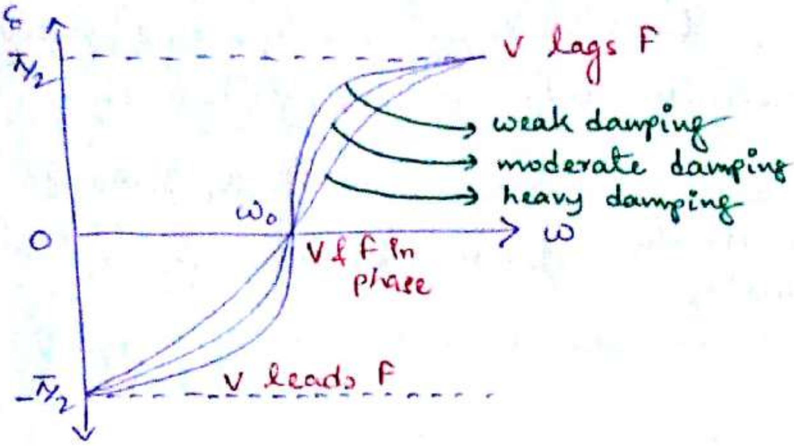
$V_0^{max} = \dfrac{F_0/m}{\gamma} = \dfrac{F_0}{b}$, so as "b" increases, V_0^{max} decreases.

for $\omega >> \omega_0$, $V_0 \simeq \dfrac{F_0}{m\omega}$ and if γ is not large then $V_0 \to 0$ for $\omega \to \infty$

for $\omega << \omega_0$, $V_0 \simeq \dfrac{F_0}{m\omega_0^2} = \dfrac{F_0}{m\omega}\dfrac{\omega^2}{\omega_0^2} \to 0$ for $\omega \to \infty$.



Phase of velocity relative to the force is $\phi = \delta - \pi/2$. For $\omega << \omega_0$, $\delta \simeq 0$, so $\phi = -\pi/2$. As ϕ is angle by which velocity lags behind the force, so here velocity leads the force by an angle $\pi/2$. For $\omega >> \omega_0$, $\delta \simeq \pi$, $\phi = \pi - \pi/2 = \pi/2$ so for very high frequencies, velocity lags the force by $\pi/2$. At resonance $\omega = \omega_0$, $\delta = \pi/2$ and $\phi = 0$ & velocity is in phase with force.



This is therefore the most favourable condition for transfer of energy from the external periodic force to the oscillator.

Power transfer from driving force to the oscillator

Energy of a damped oscillator decreases exponentially as $E(t) = E_0 e^{-\gamma t}$. In order to maintain steady state oscillation, driving force transfers energy to oscillator. Now

$$x = B\cos(\omega t - \delta) = B\cos\delta\cos\omega t + B\sin\delta\sin\omega t$$

$$= B_{el}\cos\omega t + B_{ab}\sin\omega t$$

where B_{el} = elastic amplitude $B\cos\delta = \dfrac{f_0(\omega_0^2-\omega^2)}{(\omega_0^2+\omega^2)^2+\gamma^2\omega^2}$ [in phase with force]

B_{ab} = absorptive amplitude $B\sin\delta = \dfrac{f_0\,\omega\gamma}{(\omega_0^2-\omega^2)^2+\gamma^2\omega^2}$ [out of phase $\pi/2$ with force]

$v = \dot{x} = \omega(-B_{el}\sin\omega t + B_{ab}\cos\omega t)$ & thus the power by driving force $F_0\cos\omega t$ / second is the work done by the force/second

$$P(t) = F_0\cos\omega t\ v = F_0\,\omega\cos\omega t\,(-B_{el}\sin\omega t + B_{ab}\cos\omega t).$$

∴ Time averaged input power over one complete cycle is

$$P_{input} = \langle P(t)\rangle = \frac{1}{T}\int_0^T P(t)\,dt = -F_0\,\omega\frac{B_{el}}{T}\int_0^T \underbrace{\sin(\omega t)\cos(\omega t)\,dt}_{0} + F_0\,\omega\frac{B_{ab}}{T}\int_0^T \underbrace{\cos^2(\omega t)\,dt}_{\to 1/2} = \frac{1}{2}F_0\,\omega B_{ab}.$$

This input power supplied by driving force is not stored in oscillator but dissipated as work done in moving the system against friction. Instantaneous power dissipated through friction is

$$P(t) = bv\cdot v = b\left(\frac{dx}{dt}\right)^2 = b\omega^2(B_{ab}^2\cos^2\omega t + B_{el}^2\sin^2\omega t - 2B_{ab}B_{el}\cos\omega t\sin\omega t)$$

$\therefore$ Time averaged power $\langle P(t)\rangle = P_{dissipation} = \frac{b\omega^2}{2}(B_{el}^2 + B_{ab}^2)$.

$$= \frac{b\omega^2 f_0^{\,2}}{2[(\omega_0^2-\omega^2)^2 + \omega^2\gamma^2]} = \frac{1}{2}F_0\,\omega B_{ab}$$

$\therefore P_{input} = P_{dissipate}$ (steady state).

<u>Energy of the forced oscillator</u> Instantaneous KE is

$$\frac{1}{2}mv^2 = \frac{1}{2}m\omega^2(B_{ab}^2\cos^2\omega t + B_{el}^2\sin^2\omega t - 2B_{ab}B_{el}\cos\omega t\sin\omega t)$$

Instantaneous PE $\frac{1}{2}Kx^2 = \frac{1}{2}m\omega_0^2(B_{ab}^2\sin^2\omega t + B_{el}^2\cos^2\omega t + 2B_{ab}B_{el}\cos\omega t\sin\omega t)$

$\therefore$ Time averaged total energy is $E = \langle E(t)\rangle = \frac{1}{4}m(\omega^2+\omega_0^2)(B_{ab}^2 + B_{el}^2)$

$E_{resonance} = \frac{1}{2}m\omega_0^2(B_{ab}^2 + B_{el}^2)$ at $\omega \simeq \omega_0$

$\langle KE\rangle = \frac{1}{4}m\omega^2(B_{ab}^2 + B_{el}^2)$, $\langle PE\rangle = \frac{1}{4}m\omega_0^2(B_{ab}^2 + B_{el}^2)$

<u>Maximum input power & Bandwidth</u>

Time averaged input power $P_{input} = \frac{1}{2}F_0\,\omega B_{ab}$

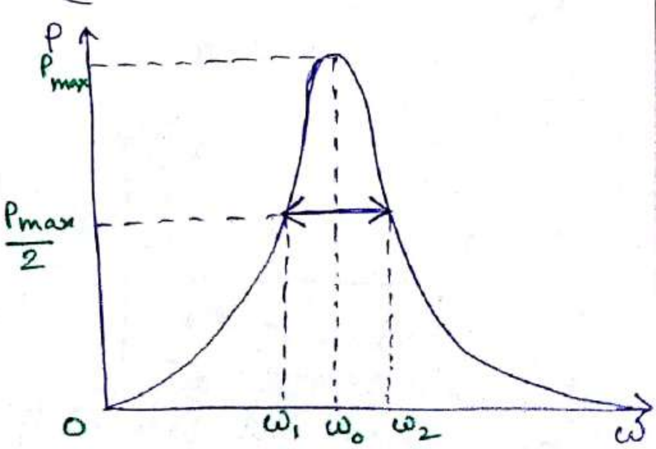
$$= \frac{F_0^2\gamma}{2m}\left[\frac{\omega^2}{(\omega_0^2-\omega)^2 + \gamma^2\omega^2}\right]$$

This will be maximum for $\frac{dP}{d\omega} = 0$

& that yields $\omega = \omega_0$. Thus at resonance frequency P_{input} is maximum.

$$P_{input}^{max} = \frac{F_0^2}{2m\gamma} \qquad \therefore \quad P = P_{input}^{max}\frac{\gamma^2\omega^2}{(\omega_0^2-\omega^2)^2 + \gamma^2\omega^2}$$

Frequency ω_1 & ω_2 at which the power drops down to 1/2 of maximum is the half power freq.



$$\frac{1}{2} = \frac{P_{input}}{P_{input}^{max}} = \frac{\gamma^2\omega^2}{(\omega_0^2-\omega^2)^2 + \gamma^2\omega^2}$$

or $\omega^2 = \omega_0^2 \pm \gamma\omega$

$$\begin{cases}\omega_1 = -\frac{\gamma}{2} + \left(\omega_0^2 + \frac{\gamma^2}{4}\right)^{1/2} \\ \omega_2 = \frac{\gamma}{2} + \left(\omega_0^2 + \frac{\gamma^2}{4}\right)^{1/2}\end{cases}$$

, band width $\Delta\omega = \omega_1 - \omega_2 = \gamma$.

Quality Factor Q is a parameter that gives the sharpness of resonance & defined as $Q = \frac{\text{resonant frequency}}{\text{band width}} = \frac{\omega_0}{\Delta\omega} = \frac{\omega_0}{\gamma}$

$$= 2\pi \frac{\text{Avg. energy stored in one cycle}}{\text{Avg. energy lost in one cycle}}$$

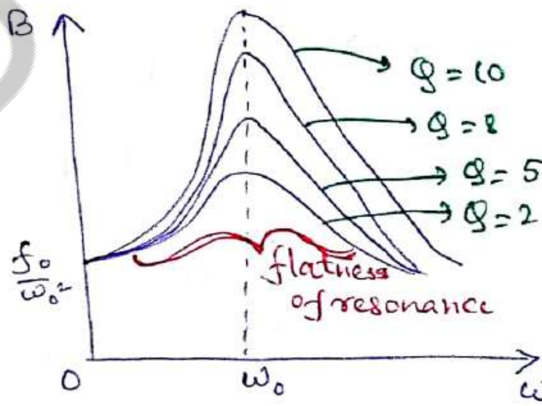
$$\therefore Q = 2\pi \frac{\langle E(t)\rangle}{P_{dissipate}\, T} = \left(\frac{2\pi}{T}\right)^{=\omega} \frac{1}{\cancel{4}_{\,2}} m(\omega^2+\omega_0^2)\cancel{(B_{ab}^2+B_{el}^2)} \frac{\cancel{2}}{b\omega^{\cancel{2}}\cancel{(B_{ab}^2+B_{el}^2)}}$$

$$= \frac{\omega^2+\omega_0^2}{2\gamma\omega} \quad \text{and for } \omega \approx \omega_0,\ Q^{resonance} = \frac{\omega_0}{\gamma}$$

Thus for low damping, $\gamma << \omega_0$ and Q is high. that makes the resonance very ~~high~~. sharp. Thus Q measures the sharpness of resonance.

Using $Q = \frac{\omega_0}{\gamma}$, the amplitude is

$$B = \frac{f_0 Q}{\omega\omega_0\sqrt{1+Q^2\left(\frac{\omega_0}{\omega}-\frac{\omega}{\omega_0}\right)^2}}$$



Q large, B large. Q can be regarded as amplification factor. at low driving force $\omega \to 0$, $B_0 = \frac{F_0/m}{\sqrt{(\omega_0^2-\omega^2)^2+\omega^2\gamma^2}} \simeq \frac{f_0}{\omega_0^2}$ and we know

$$B_{max} = \frac{f_0}{\gamma\sqrt{\omega_0^2-\gamma^2/4}}. \quad \text{So} \quad \frac{B_{max}}{B_0} = \frac{\omega_0^2}{\gamma\sqrt{\omega_0^2-\gamma^2/4}} = \frac{Q}{\sqrt{1-\frac{1}{4Q^2}}}$$

(for low damping) $= Q\left(1-\frac{1}{4Q^2}\right)^{-1/2} \simeq Q\left(1+\frac{1}{8Q^2}\right)$

Q is very large

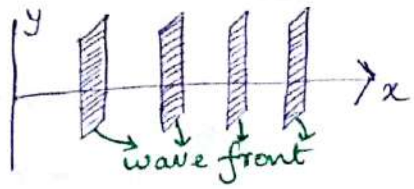
$$= Q.$$

$\therefore \boxed{B_{max} = Q B_0}$ The resonant amplitude is Q times the amplitude at low frequencies of the driving force.

Wave Motion

Equation of a progressive wave propagating through a continuous medium where the particles of the medium execute SHM is

$$y = A\sin(\omega t - kx) = A\sin\frac{2\pi}{\lambda}(vt - x)$$



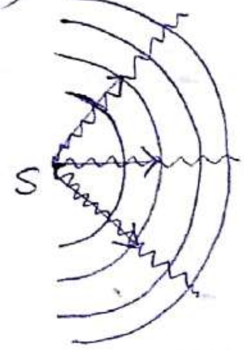
for propagation along +ive x-direction.

Similarly $y = A\sin(\omega t + kx) = A\sin\frac{2\pi}{\lambda}(vt + x)$

for propagation along −ive x-direction.

Note that if t increases by Δt and x by $v\Delta t$, then y will be restored to initial value. Thus a "disturbance" at one place is repeated at a position $v\Delta t$ after time Δt, with propagating velocity v, that propagates without damping. In reality, its amplitude gradually diminishes due to resistive forces of the viscous medium.

Equation of such wave is $y = Ae^{-\gamma x}\sin\frac{2\pi}{\lambda}(vt - x)$ which is plane progressive wave.

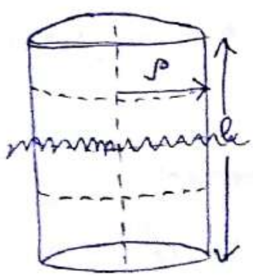


If the wave is diverging then without any dissipation of energy (no damping), the amplitude falls of. In case of ~~spherical~~ spherical progressive wave, Intensity $I \propto \frac{W}{4\pi r^2}$

where W is the energy emitted from source. $\therefore$ Amplitude$^2 \propto \frac{1}{r^2}$

or Amplitude $\propto 1/r$. Such wave is represented as

$$y = \frac{A}{|r|}\sin\frac{2\pi}{\lambda}(vt - |r|) \qquad |r| = \sqrt{x^2+y^2+z^2} \text{ (Expanding wave)}$$



For cylindrical progressive wave $I = \frac{W}{2\pi\rho l}$ or $I \propto \frac{1}{\rho}$

$\therefore A^2 \propto \frac{1}{\rho}$ or $A \propto \frac{1}{\sqrt{\rho}}$

Such wave is represented by $y = \frac{A}{\sqrt{\rho}}\sin\frac{2\pi}{\lambda}(vt - \rho)$

The velocity $v = \dot{x}$ is called the wave velocity as the disturbance propagates with this velocity. This is also called phase velocity as during motion of the wave, phase of the motion of the particles move with this velocity. But as the wave move onward, particles vibrate about their mean position of rest which is called the particle velocity. V

Now $y = A\sin(\omega t - kx)$ gives $V = \frac{dy}{dt} = A\omega\cos(\omega t - kx)$

and $\frac{dy}{dx}$ = slope of the displacement of the particle

$$= -AK\cos(\omega t - kx) = -AK\frac{V}{A\omega} = -\frac{K}{\omega}V$$

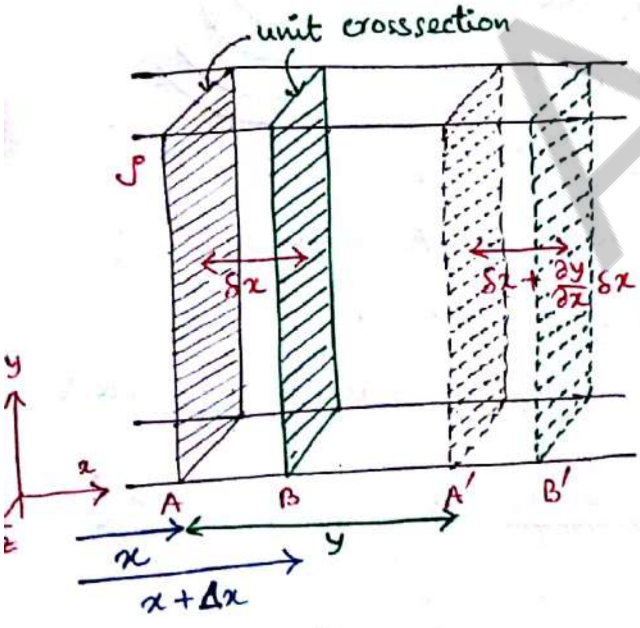
$$\therefore \boxed{\underbrace{V}_{\text{particle velocity}} = -\frac{\omega}{K}\frac{dy}{dx} = -\underbrace{v}_{\text{phase velocity}}\frac{dy}{dx}}$$ → negative gradient of displacement.

Also, $\frac{\partial^2 y}{\partial t^2} = -A\omega^2\sin(\omega t - kx)$, $\frac{\partial^2 y}{\partial x^2} = -AK^2\sin(\omega t - kx)$

$$\therefore \frac{\partial^2 y}{\partial x^2} = \left(\frac{K}{\omega}\right)^2\frac{\partial^2 y}{\partial t^2} = \frac{1}{v^2}\frac{\partial^2 y}{\partial t^2}$$

$$\therefore \boxed{\frac{\partial^2 y}{\partial t^2} = v^2\frac{\partial^2 y}{\partial x^2}}$$ is the differential equation for a progressive wave.

Plane longitudinal wave through an elastic fluid medium



Transverse wave cannot resist any shearing stress just like a fluid. So no transverse wave is formed in a fluid. Consider a unit crosssectional area in a fluid medium with its axis in the direction of propagation of the sound wave. Let A & B are two normal planar sections at x & $x+\Delta x$ from some arbitrary origin.

In progression, let at an instant plane A is displaced by y to A' and at the same time the plane B is displaced to B' by an amount $y + \frac{\partial y}{\partial x}\delta x$.

∴ The actual position of A' plane is $(x+y)$ & B' plane is $(x + y + \delta x + \frac{\partial y}{\partial x}\delta x)$.

Considering the tube has unit cross section, volume of fluid between A & B is δx & that between A' & B' is $\delta x + \frac{\partial y}{\partial x}\delta x$. ∴ Change in volume due to displacement $\Delta V = \frac{\partial y}{\partial x}\delta x$ & hence the volume strain $\frac{\Delta V}{V} = \frac{\frac{\partial y}{\partial x}\delta x}{\delta x} = \frac{\partial y}{\partial x}$.

If δP be the excess pressure over the normal pressure on plane A that is transferred to A', then the Bulk modulus of the medium is

$B = -\frac{\delta P}{\Delta V/V} = -\frac{\delta P}{\frac{\partial y}{\partial x}}$, or $\delta P = -B\frac{\partial y}{\partial x}$. Negative sign indicating that the volume decreases with the increase in pressure.

Now the excess pressure over face B which is transferred to B' is

$$\delta P + \frac{\partial(\delta P)}{\partial x}\delta x = -B\frac{\partial y}{\partial x} + \frac{\partial}{\partial x}\left(-B\frac{\partial y}{\partial x}\right)\delta x$$

$$= -B\frac{\partial y}{\partial x} - B\frac{\partial^2 y}{\partial x^2}\delta x \qquad [B \neq B(x)]$$

∴ Excess pressure on the volume element is

$-B\frac{\partial y}{\partial x} - \left(-B\frac{\partial y}{\partial x} - B\frac{\partial^2 y}{\partial x^2}\delta x\right) = B\frac{\partial^2 y}{\partial x^2}\delta x$. This pressure will create an accelaration along the direction of force within the volume element. If ρ is the density (medium) then using Newton's 2nd law

$$\rho\,\delta x\frac{\partial^2 y}{\partial t^2} = B\frac{\partial^2 y}{\partial x^2}\delta x \quad \text{or} \quad \boxed{\frac{\partial^2 y}{\partial t^2} = \frac{B}{\rho}\frac{\partial^2 y}{\partial x^2} = v^2\frac{\partial^2 y}{\partial x^2}}$$

Compressional wave due to longitudinal vibrations of a long thin rod:

Consider two normal crosssection A & B at a distance x & $x + \delta x$ from some arbitrary origin on a uniform thin rod whose length is large compared to its lateral dimension. Due to flow of compressional wave along the length, A is displaced to A' at a distance y & B is displaced to B' at a distance $(y + \frac{\partial y}{\partial x}\delta x)$. The actual position of A' and B' are $(x+y)$ and $(x+\delta x + y + \frac{\partial y}{\partial x}\delta x)$. Thickness of the slice AB is δx and after time t, thickness of A'B' is $\delta x + \frac{\partial y}{\partial x}\delta x$.

∴ Change in length of the slice is $\frac{\partial y}{\partial x}\delta x$ & hence the longitudinal strain $= \dfrac{\frac{\partial y}{\partial x}\delta x}{\delta x} = \frac{\partial y}{\partial x}$. If F is the stretching force per unit crosssectional area (or stress) of the plane A & Y is the Young's modulus of the material then $Y = \dfrac{F}{\partial y/\partial x}$, ∴ $F = Y\frac{\partial y}{\partial x}$.

The stretching stress acting on B which is displaced to B′ is

$$Y\frac{\partial y}{\partial x} + \frac{\partial}{\partial x}\left(Y\frac{\partial y}{\partial x}\right)\delta x = Y\frac{\partial y}{\partial x} + Y\frac{\partial^2 y}{\partial x^2}\delta x$$

∴ The total force on AB in the +x direction is $Y\frac{\partial^2 y}{\partial x^2}\delta x$.

If ρ is material density, then $\rho\,\delta x\frac{\partial^2 y}{\partial t^2} = Y\frac{\partial^2 y}{\partial x^2}\delta x$

$$\therefore \boxed{\frac{\partial^2 y}{\partial t^2} = \frac{Y}{\rho}\frac{\partial^2 y}{\partial x^2} = v^2\frac{\partial^2 y}{\partial x^2}} \qquad v = \sqrt{\frac{Y}{\rho}}$$

For iron, $Y = 10^{12}$ dyne/cm², $\rho = 8.9$ gm/cc. $v \simeq 10^6$ cm/s.

If lateral strain is also accounted then, $v = \sqrt{\dfrac{Y + \frac{4}{3}\eta}{\rho}}$, where η = modulus of rigidity. This indicates that consideration of lateral strain enhances the velocity of longitudinal wave through the medium.

Longitudinal waves in a Gas $\qquad \frac{\partial^2 y}{\partial t^2} = \frac{E}{\rho}\frac{\partial^2 y}{\partial x^2}$, $v = \sqrt{\frac{E}{\rho}}$

where E is volume elasticity (bulk modulus) of gas, ρ = density.

Newton's Formula : Newton first calculated the velocity of sound wave in a gas, on the assumption that temperature variation is negligible, so in isothermal process, Boyle's law is applicable

PV = constant or $P\delta V + \delta P\, V = 0$ or $P = -\dfrac{\delta P}{\delta V/V} = E$

∴ $v = \sqrt{\frac{E}{\rho}} = \sqrt{\frac{P}{\rho}}$. For air at standard pressure & temperature (STP), $\rho = 1.29$ kg/m³. $P = 0.76$m of Hg $= 0.76 \times 13.6 \times 10^3 \times 9.8$ N/m²

$v = 280$ m/s

But experimental value came as $v = 332$ m/s.

Laplace's correction: Temperature correction, region of compression is heated & region of rarefaction is cooled. Since the thermal conductivity of a gas is small, that these thermal changes occur at much faster time scale that heat developed during compression & cooling due to rarefaction is not transferred out to thermalize. So in adiabatic condition, $PV^{\gamma} = \text{constant}$, $\gamma = C_P/C_V$

$$\therefore \Delta P V^{\gamma} + \gamma P V^{\gamma-1} \Delta V = 0 \quad \text{or} \quad \gamma P = -\frac{\Delta P V^{\gamma}}{V^{\gamma-1}\Delta V} = -\frac{\Delta P}{\Delta V/V} = E.$$

$\therefore v = \sqrt{\frac{\gamma P}{\rho}}$. $\gamma = 1.4$ for air at STP, $v = 331.6$ m/S.

Now $PV = \frac{m}{M} RT$ and $\rho = \frac{m}{V}$ $\therefore v = \sqrt{\frac{\gamma RT}{M}}$.

So velocity is independent of pressure or density, $v \propto \sqrt{T}$, $v \propto \frac{1}{\sqrt{M}}$, $v \propto \sqrt{\gamma}$ i.e. whether monoatomic, diatomic gas etc.

Energy Transport in Travelling Waves

When a plane progressive harmonic wave passes through a medium, medium particles contain extra energy due to SHM in terms of KE & PE. Total energy is conserved.

$y = a \sin \frac{2\pi}{\lambda}(vt - x)$ is displacement

velocity of each particle $\mathbf{V} = \frac{dy}{dt} = \frac{2\pi v a}{\lambda} \cos \frac{2\pi}{\lambda}(vt - x)$

For ρ = density, KE per unit volume $= \frac{1}{2}\rho V^2 = \frac{1}{2}\rho\left(\frac{2\pi v a}{\lambda}\right)^2 \times \cos^2 \frac{2\pi}{\lambda}(vt - x)$

Since this varies over time, average KE density of the medium is $\langle KE \rangle = \frac{\rho}{2} \frac{4\pi^2 v^2 a^2}{\lambda^2} \underbrace{\langle \cos^2 \frac{2\pi}{\lambda}(vt - x) \rangle}_{= 1/2} = \frac{\rho \pi^2 v^2 a^2}{\lambda^2}$

To evaluate the average P.E. we calculate work done for the decrease of volume δV against an average pressure $\frac{P + (P + \delta P)}{2}$ is

$dW = \left(P + \frac{\delta P}{2}\right)\delta V$. But $B = -\frac{\delta P}{\delta V/V}$ and $\delta V = -\frac{\partial y}{\partial x}\delta x$

$\therefore \delta P = -\frac{B}{V}\delta V = -B\frac{\partial y}{\partial x}$

$= \left(P - \frac{B}{2}\frac{\partial y}{\partial x}\right)\left(-\frac{\partial y}{\partial x}\delta x\right)$

as $\delta V/V = \frac{\partial y}{\partial x}$.

$$= -P\frac{\partial y}{\partial x}\,\delta x + \frac{B}{2}\left(\frac{\partial y}{\partial x}\right)^2 \delta x$$

$\therefore$ The work done per unit volume $W = -P\frac{\partial y}{\partial x} + \frac{B}{2}\left(\frac{\partial y}{\partial x}\right)^2$

$$= P\frac{2\pi a}{\lambda}\cos\frac{2\pi}{\lambda}(vt-x) + \frac{B}{2}\left(\frac{2\pi a}{\lambda}\right)^2 \cos^2\frac{2\pi}{\lambda}(vt-x)$$

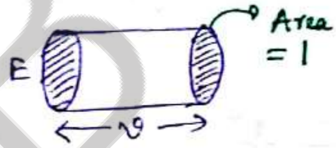
$\therefore$ Time averaged P.E. is $\langle P.E.\rangle = \frac{2\pi a P}{\lambda}\underbrace{\cancel{\langle \cos\frac{2\pi}{\lambda}(vt-x)\rangle}}_{\to 0} + \frac{4\pi^2 a^2 B}{2\lambda^2}\underbrace{\langle \cos^2\frac{2\pi}{\lambda}(vt-x)\rangle}_{1/2}$

$$= \frac{\pi^2 a^2 B}{\lambda^2} \qquad \text{as } v = \sqrt{B/\rho}$$

$$\langle PE\rangle = \frac{\pi^2 \rho v^2 a^2}{\lambda^2} = \langle KE\rangle.$$

So average value of PE & KE are same. Therefore total energy of the medium is $\boxed{E = \frac{2\pi^2 \rho v^2 a^2}{\lambda^2}}$ which is the energy crossing unit area per unit time or intensity. $I \propto a^2, \propto \rho, \propto 1/\lambda^2, \propto v^3$

$$I = vE = \frac{2\pi^2 \rho v^3 a^2}{\lambda^2}$$



Unit of Intensity: Bel & Decibel

The ratio of sound intensity from low to high in the detectable range is $1:10^{14}$. So logarithmic ratio is $14:1$ for high to low. For all practical purposes, absolute intensity is unnecessary & relative values have more practical significance.

Bel & Decibel are logarithmic units of relative intensity. If ratio of sound intensity is $10:1$ then difference of intensity is "1 Bell." $N = \log_{10} I_1/I_2$ where N = number of bels and I_1 & I_2 are intensity of two sounds. A "decibel (db)" is $\frac{1}{10}$th of Bel. $n = 10\log_{10} I_1/I_2$ where n = no. of decibels.

1 bel $= 10$ dB $= 10^1:1$ $\qquad \therefore 1\,dB = 10^{0.1}:1 \simeq 1.26:1$

$\therefore N$ bels $= 10N$ dB $= 10^N:1$

$0.1n$ bels $= n$ dB $= 10^{0.1n}:1$

Intensity level (IL) IL is ratio of intensity to standard intensity I_0 where threshold is 10^{-12} watt/m^2 or 10^{-16} watt/cm^2 which corresponds to lower limit of intensity for audability.

$$\therefore \quad (IL)_{bel} = \log_{10}\left(\frac{I}{I_0}\right), \quad (IL)_{db} = 10\log_{10}\left(\frac{I}{I_0}\right)$$

This means $I = 10^2 I_0$, $(IL)_{bel} = 2$ bels $= 20$ decibels

$I_{max} = 10^{14} I_0$, $(IL)_{bel} = 14$ bels $= 140$ decibels.

Conversely intensity 140 db means $140 = 10\log_{10}\frac{I}{I_0}$

or $I = I_0 \times 10^{14} = 10^{-16} \times 10^{14} = 10^{-2}$ watt/cm^2

Relation between wave intensity & mean square of excess pressure

The excess pressure due to propagation of sound wave is

$$p = -B\frac{\partial y}{\partial x} \quad \text{where } B = \rho v^2 = \text{Bulk modulus of the medium}$$

displacement $y = a\sin\frac{2\pi}{\lambda}(vt - x)$

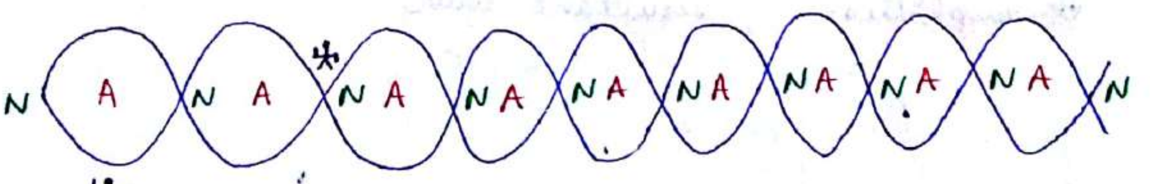
$$\therefore p = v^2 \rho a \frac{2\pi}{\lambda}\cos\frac{2\pi}{\lambda}(vt-x) \quad \text{or} \quad p^2 = \frac{4\pi^2 v^4 \rho^2 a^2}{\lambda^2}\cos^2\frac{2\pi}{\lambda}(vt-x)$$

$$\text{or} \quad \langle p^2 \rangle = \frac{4\pi^2 v^4 \rho^2 a^2}{\lambda^2}\left\langle \cos^2\frac{2\pi}{\lambda}(vt-x)\right\rangle = \frac{2\pi^2 v^4 \rho^2 a^2}{\lambda^2}$$

But we know $I = \frac{2\pi^2 v^3 \rho a^2}{\lambda^2} = \frac{\langle p^2\rangle}{\rho v}$

So whenever pressure is maximum, I is maximum $\rightarrow$ loud sound (node)

pressure is minimum, I is minimal $\rightarrow$ very low sound (antinode)



$y \propto \sin(\;)$, $p \propto \cos(\;)$ $\therefore$ p and y are $\frac{\lambda}{2}$ phase different

so whenever y is minimum, p is maximum and vice versa.

Phase velocity and Group velocity

A single progressive wave along +ive x axis is represented by $x = a\sin(\omega t - Kx) = a\sin\frac{2\pi}{\lambda}(vt - x)$ where v is the "**phase velocity**" of the wave, as with this velocity phase of a wave moves from one point to another point.

But when two or more such harmonic waves of slightly different frequencies are superposed, the anharmonic phenomena of beat occurs. These beats are generally known as **modulations** and such anharmonic motion has modulated amplitude that repeats at a frequency called beat frequency or modulation frequency. It carries energy from one point to another with a velocity different from those of the harmonic waves. These travelling modulations that consist of group of harmonic waves are called wave packets or wave groups. The velocity with which this modulated amplitude moves is called "**group velocity**."

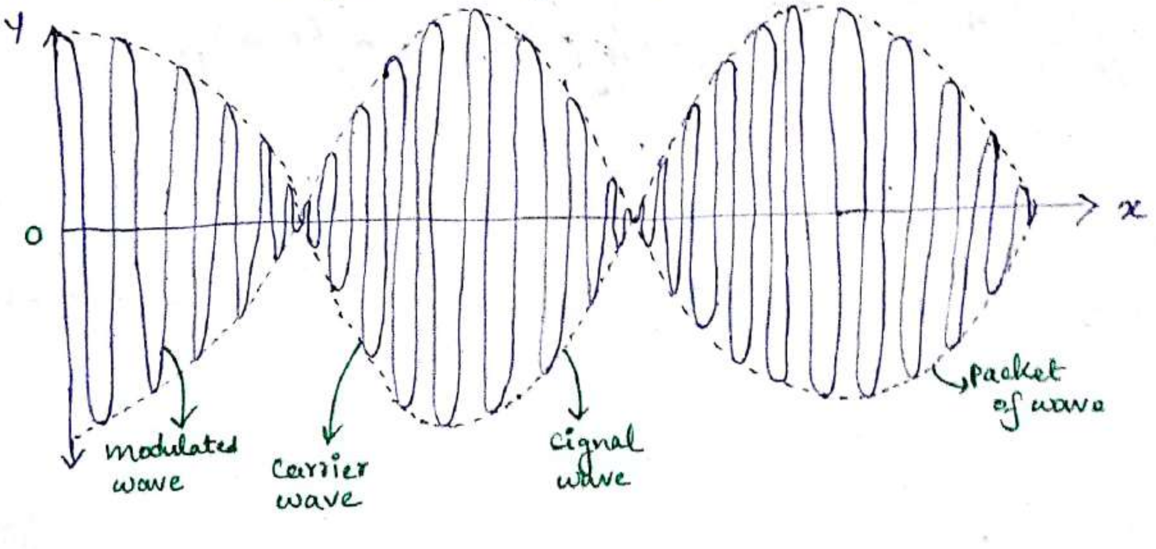
If we superpose two different waves that are slightly different in frequency/wavelength, then resultant amplitude is

$$y = y_1 + y_2 = a\sin(\omega t - Kx) + a\sin[(\omega + \Delta\omega)t - (K + \Delta K)x]$$

or $$y = \underbrace{2a\cos\left(\frac{\Delta\omega}{2}t - \frac{\Delta K}{2}x\right)}\sin\left[\left(\omega + \frac{\Delta\omega}{2}\right)t - \left(K + \frac{\Delta K}{2}\right)x\right]$$

or $$y = A\sin\left[\left(\omega + \frac{\Delta\omega}{2}\right)t - \left(K + \frac{\Delta K}{2}\right)x\right]$$

$\Downarrow$ amplitude of resultant wave



This represents a travelling wave at a frequency $\omega + \frac{\Delta\omega}{2}$, wave vector $K + \frac{\Delta K}{2}$ and amplitude $2a\cos\left(\frac{\Delta\omega}{2}t - \frac{\Delta K}{2}x\right)$ which is modulated by an envelope of frequency $\Delta\omega$. Thus we can say that velocity of this amplitude is $\frac{\Delta\omega}{\Delta K}$ and in $\lim_{\Delta K \to 0}$, we get $V_g = \frac{d\omega}{dK}$ is the group velocity. Note that the phase velocity is $V_p = \frac{\omega}{K}$.

$$\therefore \frac{d\omega}{dK} = \frac{d}{dK}(KV_p) \quad \text{or} \quad V_g = V_p + K\frac{dV_p}{dK}$$

Using $K = \frac{2\pi}{\lambda}$, $dK = -\frac{2\pi}{\lambda^2}d\lambda$, $V_g = V_p - \frac{2\pi}{\lambda}\frac{\lambda^2}{2\pi}\frac{dV_p}{d\lambda}$

$$\text{or} \quad \boxed{V_g = V_p - \lambda\frac{dV_p}{d\lambda}}$$

<u>Dispersive Medium</u> A medium where the phase velocity of any wave varies with frequency/wavelength is called a dispersive medium. Glass, water & all transparent substances are dispersive for electromagnetic waves. In dispersive medium $\lambda_1 \neq \lambda_2 \neq \lambda_3 \neq \cdots \neq \lambda_n$ and V_g is always greater than V_p. However in nondispersive medium $\lambda_1 = \lambda_2 = \cdots = \lambda_n$ and $V_g = V_p$. Vaccum is nondispersive to EM waves but dispersive for deBroglie wave (matter wave). Note that all mediums are nondispersive for sound waves, therefore all waves of different wavelength move with a constant velocity in a medium.

$$\text{Now} \quad \frac{1}{V_g} = \frac{dK}{d\omega} = \frac{d}{d\omega}\left(\frac{\omega}{V_p}\right) = \frac{1}{V_p} - \frac{\omega}{V_p^2}\frac{dV_p}{d\omega}$$

As $\omega = 2\pi\nu$, $d\omega = 2\pi d\nu$. $\quad \therefore \frac{1}{V_g} = \frac{1}{V_p} - \frac{\nu}{V_p^2}\frac{dV_p}{d\nu}$

But $V_p = \frac{c}{n}$ where n is the refractive index of the medium.

$$\therefore \frac{1}{V_g} = \frac{1}{V_p} - \frac{\nu}{(c/n)^2}\frac{d}{d\nu}\left(\frac{c}{n}\right) = \frac{1}{V_p} + \frac{\cancel{n^2}\nu}{c^{\cancel{2}}}\frac{\cancel{c}}{\cancel{n^2}}\frac{dn}{d\nu} = \frac{1}{V_p} + \frac{1}{\lambda}\frac{dn}{d\nu}$$

$\left[\text{as } c = \nu\lambda \text{ or } \nu = \frac{c}{\lambda} \text{ or } d\nu = -\frac{c}{\lambda^2}d\lambda.\right] \quad = \frac{1}{V_p} - \frac{\lambda}{c}\frac{dn}{d\lambda}$

$$\therefore V_g = \left(\frac{1}{V_p} - \frac{\lambda}{c}\frac{dn}{d\lambda}\right)^{-1} = \left(\frac{1}{V_p}\right)^{-1}\left(1 - \frac{\lambda V_p}{c}\frac{dn}{d\lambda}\right)^{-1} \simeq V_p\left(1 + \frac{\lambda}{n}\frac{dn}{d\lambda}\right)$$

$$\therefore \boxed{V_g = V_p\left(1 + \frac{\lambda}{n}\frac{dn}{d\lambda}\right)}$$

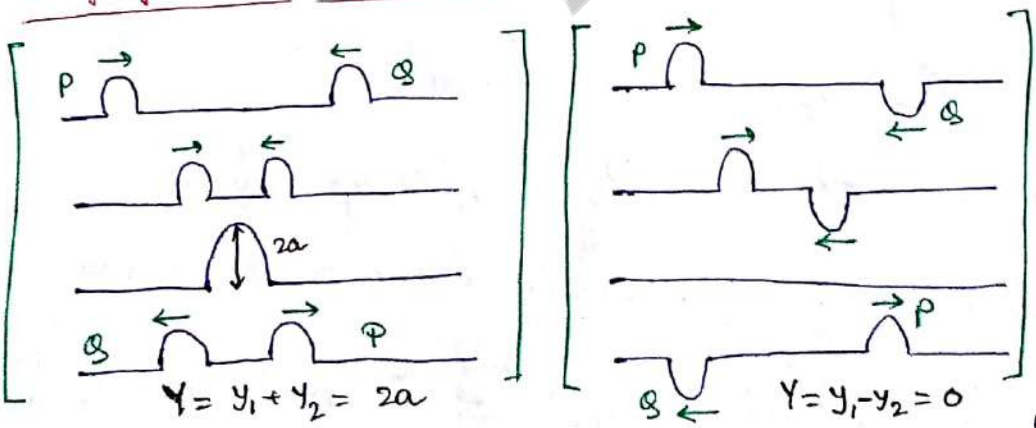
Loudness of Sound : Phon & Sone

Intensity of sound refers to a purely physical aspect of sound which is the amount of energy reaching unit area in unit time. But the loudness of sound is a subjective phenomena & it refers to the sensation produced to ear. Sensation of loudness depends on the intensity of sound, but loudness is not proportional to intensity. According to Waber & Fechner, $\delta L \propto \frac{\delta I}{I}$ where δL is increment in loudness & δI is that of intensity.

$\therefore$ $\delta L = K\frac{\delta I}{I}$ or $L = K \ln I$ where K is some constant.

The absolute unit of intensity is Watt/cm^2 but it is expressed is decibel. The unit of loudness is "**phon**." To measure loudness of sound in phons, another pure tone of frequency 1000 cycles per second (C.P.S.) is taken whose intensity is gradually altered till its loudness becomes exactly equal to that of the given sound. If then the intensity level of the pure tone in n decibel, then the loudness of that sound is n phon. Phon is a small unit, another larger unit is "**sone**" which is equal to 40 times of phon.

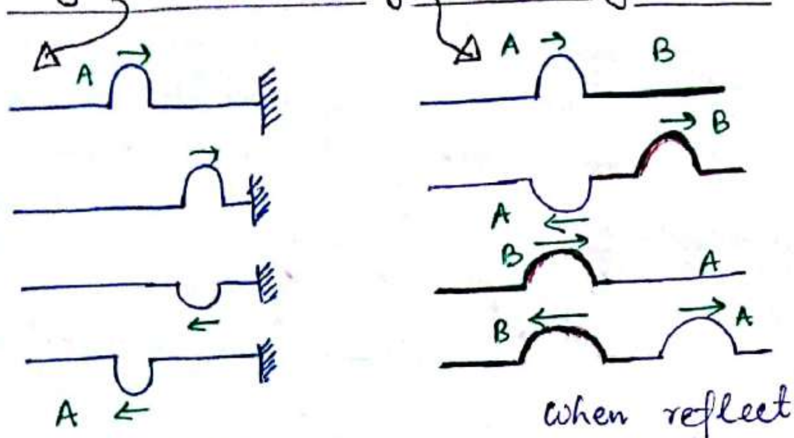
Superposition of Waves



Different waves can pass through each other simultaneously through the same medium. We can hear distinctly the conversation in room, or when dropped a pebble in pond, nice ripples passing (solitonic waves satisfying KdV equation).

"Processes by which different trains of waves travelling through a medium simultaneously overlap into one another without losing their individual nature/shape is called superposition of waves."

Reflection & refraction of waves



rarer to denser

denser to rarer

$y_p = a \sin \frac{2\pi}{\lambda}(vt - x)$

$y_r = -a \sin \frac{2\pi}{\lambda}(vt + x)$

$= a \sin\left[\frac{2\pi}{\lambda}(vt+x) \pm \pi\right]$

when reflected from denser to rarer medium, no phase change occur. But from rarer to denser reflection 180° phase change occur. λ and v of refracted wave different than incident wave But ν will remain unchanged & accoding to

Snell's law, $\nu = \frac{v_i}{\lambda_i} = \frac{v_r}{\lambda_r}$

Comparison between Interference & Beats / Progressive & stationary wave

Interference	Beats
1. Two waves must have same frequency.	1. Two waves must have slightly different frequencies.
2. Phase difference of two waves is constant.	2. Phase difference varies from 0 to π with time.
3. Position of maximum & minimum intensity remains unchanged.	3. Positions of maximum & minimum intensity changes continuously.

—— X ——

Progressive wave	Stationary wave
1. Produced due to continuous periodic vibrations of medium particles.	1. Produced due to superposition of two identical progressive waves (colinear) in opposite directions.
2. Particles execute identical periodic motion about mean position.	2. Except particles at nodes, execute motions of varying amplitudes.
3. wave advances with definite velocity.	3. wave does not advance in medium.
4. wave retains its shape	4. wave change by shrinking to straight line twice in each period.

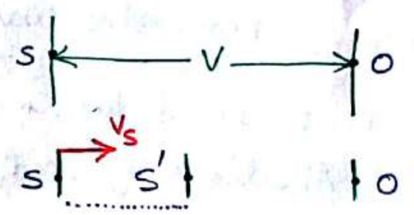
Progressive waves	Stationary waves
5. Same amplitude.	5. Amplitude varies continuously with maximum at antinode & minimum at node.
6. Phase change with space & time.	6. phase between two nodes is same & changes with time. Phase change by π from one loop to other.
7. In a complete Time period medium particle never came to rest.	7. In a complete time period, all particles come to rest twice together.

Doppler Effect

When there exist a relative motion between the source and observer, then the apparent change in frequency of the sound as perceived by the observer is known as Doppler Effect. Example: sound heard by a fast mail train by an observer at platform.

This is because if ν frequency emitted by source is received by observer then both are stationary, but if they're in motion then received frequency can be less or more. When observer approaches the source, the apparent frequency will increase because observer receives more waves / second.

Let both observer & medium is stationary (no wind) & the soure is moving uniformly towards the observer.



V = velocity of sound (in air)

V_s = velocity of source (train)

ν = frequency of emitted sound

when S is stationary $\nu = \frac{V}{\lambda}$ and $SO = V$. Now source S is movin with velocity V_s, then $SS' = V_s$ and $S'O = V - V_s$. As velocity of sound is constant, ν waves will be contained in $V - V_s$. So the decreased wavelength is $\lambda' = \frac{V - V_s}{\nu}$ & so the apparent frequency heard by observer $\nu' = \frac{V}{\lambda'} = \nu \frac{V}{V - V_s}$.

$\therefore$ Increase in frequency $\Delta\nu = \nu' - \nu = \frac{\nu V}{V - V_s} - \nu = \frac{\nu V_s}{V - V_s}$.

If S is moving away from O then, $\nu' = \nu \frac{V}{V + V_s}$ and then decrease in frequency is $\Delta\nu = \nu - \nu' = \frac{\nu V_s}{V + V_s}$.

If wind blows at V_ω towards the direction of sound then effective velocity of sound is $V + V_\omega$, then $\nu' = \nu \frac{V + V_\omega}{V + V_\omega - V_s}$. If direction is opposite then, $\nu' = \nu \frac{V - V_\omega}{V - V_\omega - V_s}$.

<u>CW</u> Two observers A & B have sources of sound of frequency 500 Hz. If A remains stationary while B moves away with velocity 10 m/s find the no. of beats heard by A and B. $V_{sound} = 332$ m/s.

<u>Beats heard by A</u> A = stationary, B = moving, "away"

$$\nu' = \nu \frac{V}{V + V_s} = 500 \times \frac{332}{332 + 10} = 485.4 \text{ Hz}$$

$\therefore$ frequency of beats heard by A = $500 - \nu' = 14.6$ Hz.

<u>Beats heard by B</u> A = moving, B = stationary,

$$\nu' = \nu \frac{V - V_o}{V} = 500 \times \frac{332 - 10}{332} = 485 \text{ Hz}.$$

$\therefore$ frequency of beats heard by B = $500 - \nu' = 15$ Hz.

Observer fixed	Source fixed
(a) Towards $\nu' = \nu \frac{V}{V - V_s}$ Away $\nu' = \nu \frac{V}{V + V_s}$	(b) Towards $\nu' = \nu \frac{V + V_o}{V}$ Away $\nu' = \nu \frac{V - V_o}{V}$
Source observer moving	
(c) $\nu' = \nu \frac{V \pm V_o}{V \pm V_s}$	